Kinetic Medians and kd-Trees

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Abstract. We propose algorithms for maintaining two variants of kd-trees of a set of moving points in the plane. A pseudo kd-tree allows the number of points stored in the two children to differ. An overlapping kd-tree allows the bounding boxes of two children to overlap. We show that both of them support range search operations in $O(n^{1/2+\epsilon})$ time, where ϵ only depends on the approximation precision. When the points move, we use event-based kinetic data structures to update the tree when necessary. Both trees undergo only a quadratic number of events, which is optimal, and the update cost for each event is only polylogarithmic. To maintain the pseudo kd-tree, we make use of algorithms for computing an approximate median level of a line arrangement, which itself is of great interest. We show that the computation of the approximate median level of a set of lines or line segments, can be done in an online fashion smoothly, i.e., there are no expensive updates for any events. For practical consideration, we study the case when there are speed limit restrictions or smooth trajectory requirements. The maintenance of the pseudo kd-tree, as a consequence of the approximate median algorithm, can also adapt to those restrictions.

1 Introduction

Motion is ubiquitous in the physical world. Several areas such as digital battlefields, air-traffic control, mobile communication, navigation system, geographic information systems, call for storing moving objects into a data structure so that various queries on them can be answered efficiently; see [24, 26] and the references therein. The queries might relate either to the current configuration of objects or to a configuration in the future — in the latter case, we are asking to predict the behavior based on the current information. In the last few years there has been a flurry of activity on extending the capabilities of existing database systems to represent moving-object databases (MOD) and on indexing moving objects; see, e.g., [15, 23, 24]. The known data structures for answering range queries on moving points either do not guarantee a worst-case bound on the query time [27, 26, 20] or are too complicated [1, 16].

In this paper we develop kinetic data structures for kd-trees, a widely used data structure for answering various proximity queries in practice [10], which can efficiently answer range queries on moving points. Originally proposed by Basch *et al.* [5], the *kinetic data structure* framework has led to efficient algorithms for several geometric

problems involving moving objects; see [14] and references therein. The main idea in the kinetic framework is that even though the objects move continuously, the relevant combinatorial structure of the data structure changes only at certain discrete times. Therefore one does not have to update the data structure continuously. The *kinetic updates* are performed on the data structure only when certain *kinetic events* occur; see [5, 14] for details.

Recall that a kd-tree on a set of points is a binary tree, each node v of which is associated with a subset S_v of points. The points in S_v are partitioned into two halves by a vertical or horizontal line at v and each half is associated with a child of v. The orientation of the partition line alternates as we follow a path of the tree. In order to develop a kinetic data structure for kd-trees, the first step is to develop a kinetic data structure for maintaining the median of a set of points moving on a line. In fact, this subroutine is needed for several other data structures. The second problem that we study in this paper is thus maintaining the median of a set of moving points.

Related work. The problem of maintaining the point of rank k in a set S of points moving on the x-axis is basically the same as computing the k-level in the arrangement of curves; the *k-level* in an arrangement of x-monotone curves is the set of edges of the arrangement that lie above exactly k curves [4]. If the points in S are moving with fixed speed, i.e., their trajectories are lines in the xt-plane, then the result by Dey [11] implies that the point of rank k changes $O(nk^{1/3})$ times; the best-known lower bound is $ne^{\Omega(\sqrt{\log k})}$ [25]. Only much weaker bounds are known if the trajectories of points are polynomial curves. Edelsbrunner and Welzl [13] developed an algorithm for computing a level in an arrangements of lines. By combining their idea with kinetic tournaments proposed by Basch *et al.* [6] for maintaining the maximum of a set of moving points on a line, one can construct an efficient kinetic data structure for maintaining the median of set of moving points.

Since no near-linear bound is known on the complexity of a level, work has been done on computing an approximate median level. A δ-approximate median-level is defined as a x-monotone curve that lies between $(1/2 - \delta)n$- and $(1/2 + \delta)n$-levels. Edelsbrunner and Welzl [13] showed that a δ-approximate median level with at most $\lceil \lambda(n)/(\delta n) \rceil$ edges, where $\lambda(n)$ is the number of vertices on the median level of the arrangement, can be computed for line arrangements. Later Matoušek [17] proposed an algorithm to obtain a δ-approximate median level with constant complexity for an arrangement of n lines. However no efficient kinetic data structure is known for maintaining an approximate median of a set of moving points.

Because of their simplicity, kd-trees are widely used in practice and several variants of them have been proposed [3, 7, 18]. It is well known that a kd-tree can answer a two-dimensional orthogonal range query in $O(\sqrt{n} + k)$ time, where k is the number of points reported. Variants of kd-trees that support insertions and deletions are studied in [19, 9]. No kinetic data structures are known for kd-trees.

Agarwal *et al.* [1] were the first to develop kinetic data structures for answer range searching queries on moving points. They developed a kinetic data structure that answers a two-dimensional range query in optimal $O(\log n+k)$ time using $O(n \log n/(\log \log n))$ space, where k is the output size. The amortized cost of a kinetic event is $O(\log^2 n)$,

and the total number of events is $O(n^2)$.[3] They also showed how to modify the structure in order to obtain a tradeoff between the query bound and the number of kinetic events. Kollios *et al.* [16] proposed a data structure for range searching among points moving on the x-axis, based on partition trees [3]. The structure uses $O(n)$ space and answers queries in $O(n^{1/2+\epsilon}+k)$ time, for an arbitrarily small constant $\epsilon > 0$. Agarwal *et al.* [1] extended the result to 2D. Unlike kinetic data structures, these data structures are time oblivious in the sense that they do not evolve over time. However all these data structures are too complex to be used in practice.

In the database community, a number of practical methods have been proposed for accessing and searching moving objects (see [27, 26, 20] and the references therein). Many of these data structures index the trajectories of points either directly or by mapping to higher dimensions. These approaches are not efficient since trajectories do not cluster well. To alleviate this problem, one can parametrize a structure such as the R-tree, which partitions the points but allows the bounding boxes associated with the children of a node to overlap. To expect good query efficiency, the areas of overlap and the areas of the bounding boxes must be small. Although R-tree works correctly even when the overlap areas are too large, the query performance deteriorates in this case. Kinetic data structure based on R-tree were proposed in [26, 21] to handle range queries over moving points. Unfortunately, these structures also do not guarantee sublinear query time in the worst case.

Our results. Let $S = \{p_1, \ldots, p_n\}$ be a set of n points in $\mathbb{R}^1$, each moving independently. The position of a point p_i at time t is given by $p_i(t)$. We use $\bar{p}_i = \bigcup_t (p_i(t), t)$ to denote the *graph* of the trajectory of p_i in the xt-space. The user is allowed to change the trajectory of a point at any time. For a given parameter $\delta > 0$, we call a point x, not necessarily a point of S, a *δ-approximate median* if its rank is in the range $[(1/2 - \delta)n, (1/2 + \delta)n]$. We first describe an off-line algorithm that can maintain a δ-approximate median of S in a total time of $O((t/(n^2\delta^2) + 1/\delta)n \log n)$, where t is the number of times two points swap position. We then show how a δ-median can be maintained on-line as the points of S move. Our algorithm maintains a point x^* on the x-axis, not necessarily one of the input points, whose rank is in the range $(1/2 \pm \delta)n$. As the input points move, x^* also moves, and its trajectory depends on the motion of input points. We show that the speed of this point is not larger than the fastest moving point, and that our algorithm can be adapted so that the trajectory of x^* is C^k-continuous for any $k \geq 0$.

Next, let S be a set of n points in $\mathbb{R}^2$, each moving independently. We wish to maintain the kd-tree of S, so that *range searching* queries, i.e., given a query rectangle R at time t, report $|S(t) \cap R|$, can be answered efficiently, Even if the points in S are moving with fixed velocity, the kd-tree on S can change $\Theta(n^2)$ times, and there are point sets on which many of these events can cause a dramatic change of the tree, i.e., each of them requires $\Omega(n)$ time to update the tree. We therefore propose two variants of kd-trees, each of which answers a range query in time $O(n^{1/2+\varepsilon} + k)$, for any constant

[3] Agarwal *et al.* [1] actually describe their data structure in the standard two level I/O model, in which the goal is to minimize the memory access time. Here we have described their performance in the standard pointer-machine model.

$\varepsilon > 0$, (k is the number of points reported), processes quadratic number of events, and spends polylogarithmic (amortized) time at each event. The first variant is called the *δ-pseudo kd-tree*, in which if a node has m points stored in the subtree, then each of its children has at most $(1/2 + \delta)m$ points in its subtree. In the second variant, called *δ-overlapping kd-tree*, the bounding boxes of the points stored in the subtrees of two children of a node can overlap. However, if the subtree at a node contains m points, then the overlapping region at that node contains at most δm points.

As the points move, both of the trees are maintained in the standard kinetic data structure framework [5]. The correctness of the tree structure is certified by a set of conditions, called *certificates*, whose failure time is precomputed and inserted into an event queue. At each certificate failure, denoted as an *event*, the KDS certification repair mechanism is invoked to repair the certificate set and possibly the tree structure as well. In the analysis of a KDS, we assume the points follow *pseudo-algebraic motions*. By this we mean that all certificates used by the KDS can switch from TRUE to FALSE at most a constant number of times. For both pseudo and overlapping kd-trees, we show that the set of certificates has linear size, that the total number of events is quadratic, and that each event has polylogarithmic amortized update cost. Dynamic insertion and deletion, is also supported with the same update bound as for an event.

The pseudo kd-tree data structure uses our δ-approximate median algorithms as a subroutine. Unlike pseudo kd-trees, the children of a node of an overlapping kd-tree store equal number of points and the minimum bounding boxes of the children can overlap. However, maintaining overlapping kd-trees is somewhat simpler, and the analysis extends to pseudo-algebraic motion of points.

2 Maintaining the Median

2.1 Off-line maintenance

Matoušek [17] used a level shortcutting idea to find a polygonal line with complexity $O(1/\delta^2)$ and proved it is a δ-approximate median. Here we extend it to the case of a set S of line segments in the plane.

We divide the plane by vertical lines l_i's into strips such that inside each strip there are at most $\delta^2 n^2/16$ intersections and at most $\delta n/4$ endpoints. In addition, we can assume that each strip either has exactly $\delta^2 n^2/16$ intersections or exactly $\delta n/4$ endpoints, otherwise we can always enlarge the strip. The number of strips is $O(t/(n^2\delta^2) + 1/\delta)$. Along each l_i, we compute the median X_i of the intersections between S and l_i. We connect adjacent medians and the polygonal line is a δ-approximate median level. Consider a strip bounded by l_i and l_{i+1}. We first delete the segments of S that have at least one endpoint inside the strip. Let $U \subseteq S$ be the set of line segments that intersect the segment X_iX_{i+1} and that lie above X_i at l_i. Similarly let $V \subseteq S$ be the set of line segments that intersect X_iX_{i+1} and that lie below X_i at l_i. $|U| = u$, $|V| = v$. Since both X_i and X_{i+1} are on the exact median level and we have deleted at most $\delta n/4$ segments, $|u - v| \leq \delta n/4$. W.l.o.g., assume $u \leq v \leq u + \delta n/4$. Notice that the intersection of a segment in U and a segment in V must be inside the strip, see Figure 1 (i). We have $uv \leq \delta^2 n^2/16$. Therefore $u \leq \delta n/4$, $v \leq \delta n/2$. Therefore

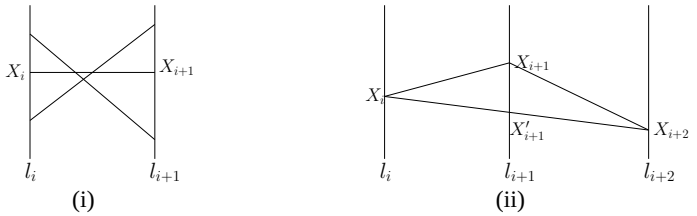


Fig. 1. (i) Approximate median level of n lines; (ii)When points change motion plan.

X_iX_{i+1} is intersected at most δn times, so it is a δ-approximation of the median level. Computing the partition line l_i involves computing the k-th leftmost intersection in an arrangement. This can be done in $O(n \log n)$ time, using the slope selection algorithm [8]. Computing a median along a cut line costs $O(n)$. So the total computational cost is $O((t/(n^2\delta^2) + 1/\delta)n \log n)$.

Theorem 1. *A δ-approximate median level of an arrangement of n line segments with at most $O(t/(n^2\delta^2) + 1/\delta)$ vertices can be found in $O((t/(n\delta^2) + n/\delta) \log n)$ time, where t is the number of intersections of the line segments.*

2.2 On-line maintenance

In previous section we assume that we know the motion plan of the points beforehand and compute the approximate median ahead of time. However, when the points change their motion plan, we want to maintain the approximate median online and distribute the amount of work more uniformly over time. Assume that the total number of motion plan changes over time is only linear in the number of points. We will show that the approximate median can be maintained smoothly, with linear number of events and polylogarithmic time update cost per event.

A deterministic algorithm. Consider the time-space plane. As like Theorem 1, the plane is divided by vertical lines l_i at time t_i into strips. We maintain the invariant that each strip either contains at most $\delta^2n^2/36$ vertices or $\delta n/3$ flight plan changes. Also if the strip contains less than $\delta n/3$ flight plan changes, then it contains at least $\delta^2n^2/72$ vertices. We approximate the median level by computing a polygonal line $X_1X_2...X_m$ in which X_i lies on l_i. We can see that the total number of strips is bounded by $O(1/\delta^2)$ since there are only linear number of flight plan changes in total.

However, instead of computing the $O(1/\delta^2)$ cut lines all at one time, we compute them one by one. We begin with computing the median on the two leftmost cut lines l_1 and l_2 (l_1 corresponds to the starting time). There are $\delta^2n^2/72$ vertices between l_1 and l_2. The approximate median then moves along line segment X_1X_2. This preprocessing costs $O(n \log n)$. Basically we compute the position of the cut line l_{i+2} and the median along l_{i+2} gradually while we are moving along X_iX_{i+1}. The position of l_{i+2} is computed such that there are $\delta^2n^2/72$ vertices between l_{i+1} and l_{i+2}. So X_{i+2} is our prediction of the median on l_{i+2}, assuming there is no flight plan change between t_{i+1}

and t_{i+2}. The cost of computing l_{i+2} and X_{i+2}, which is a slope selection problem, will be amortized over the period between t_i and t_{i+1}, using the technique proposed by Agarwal *et al.* [2]. In an online version, we need to accommodate the flight plan changes in the future. If we see $\delta n/6$ motion plan changes before we reach l_{i+1}, we start another session to compute X_{i+2} based on the current (changed) trajectories. If we reach X_{i+1} before the next delta $\delta n/6$ updates, we switch to the segment $X_{i+1}X_{i+2}$ computed from the first construction. Otherwise, if we see another $\delta n/6$ motion plan changes before we reach l_{i+1}, then we just stop and move l_{i+1} to the current position. X_{i+1} is also moved to the current position of the approximate median. The next line segment that the approximate median needs to follow is $X_{i+1}X_{i+2}$, where X_{i+2} is the value returned by the second computation. Thus inside one strip, the number of intersections is at most $\delta^2 n^2/36$, and the number of flight plan changes is at most $\delta n/3$. Next we will prove the polygonal line we get is a δ-approximation of the median level.

The only problem is, if some points change their flight plans between time t_i and t_{i+1}, then X_{i+1}, which is computed before t_i, is no longer the real median X'_{i+1}. See Figure 1 (ii). However, only the points between X_{i+1} and X'_{i+1} can cross $X_{i+1}X_{i+2}$ without crossing $X'_{i+1}X_{i+2}$. Since there are only at most $\delta n/3$ flight plan changes, there are at most $\delta n/3$ points between X_{i+1} and X'_{i+1}. Since $X'_{i+1}X_{i+2}$ is crossed by at most $2\delta n/3$ lines, $X_{i+1}X_{i+2}$ is still a δ-approximation.

Theorem 2. *Let S be a set of n points, each moving with a constant velocity. Suppose the flight paths of S change a total of m times. Then a δ-approximate median of S can be maintained in an online fashion so that the median moves continuously with at most $O(1/\delta^2)$ flight plan changes. The total maintenance cost is $O((n+m)\log n/\delta^2)$. Each flight plan change costs polylogarithmic update time.*

A randomized algorithm. In Theorem 2 finding the next cut line is based on some complex algorithms and data structures, like slope selection. Next we propose a randomized algorithm that computes the approximate median online with simple data structures.

Consider the time-space plane again, we choose a random sample S' of $\sqrt{n}$ lines and compute the arrangement $\mathcal{A}(S')$, which costs $O(n)$ [12]. From a standard probabilistic analysis, $\mathcal{A}(S')$ is a good approximation of the original arrangement $\mathcal{A}(S)$ of the n lines. We compute the uniform cutting in $\mathcal{A}(S')$. Then the expected number of vertices of $\mathcal{A}(S)$ inside the strip is $\delta^2 n^2/72$. The next cut line can be computed by finding the next $\delta^2 n/72$-th vertex of $\mathcal{A}(S')$, which costs a total of $O(\log n)$ time. If the points change motion plan inside the strip, we need to update the line arrangement $\mathcal{A}(S')$. Insertion and deletion of a line in the arrangement $\mathcal{A}(S')$ cost $O(\sqrt{n}\log n)$. Since a line is in the sample with probability $1/\sqrt{n}$, the expected update cost would be $O(\log n)$. We can get high probability by Monte Carlo methods, i.e., taking samples for at most $\log n$ times. The running time of the algorithm is $O((n+m)\log n/\delta^2)$ with high probability, where m is the total number of flight plan changes.

2.3 Approximate median with speed limit

In many applications like mobile networks and spatial databases, the maximum velocity of the points is restricted. We'll show an approximate median that moves no faster than the fastest point.

Lemma 1. *The approximate median computed above cannot move faster than the maximum speed of the n points, if the points do not change flight plans.*

Proof. Observe that the approximate median is composed of line segments connecting two median points X and Y at different time steps. So if there is a line l crossing XY and going up, there must be another line l' crossing XY and going down. So the slope of XY is bounded by the slope of l and l'. If XY is not crossed by any lines, since X and Y are both medians, they must be the same point which means that the approximate median just moves along one point. This proves the lemma.

If the points can change their flight plan, our scheme in Theorem 2 does not guarantee that the approximate median moves within the limit restriction. But we can adjust it as follows. We use X_i and X'_i to represent the precomputed and exact median respectively. Basically we let the approximate median to move towards the precomputed median, if we can get there, then everything is fine. Otherwise, we just move towards it with maximum speed possible. Figure 2 (i) shows the second case. W.l.o.g., we assume

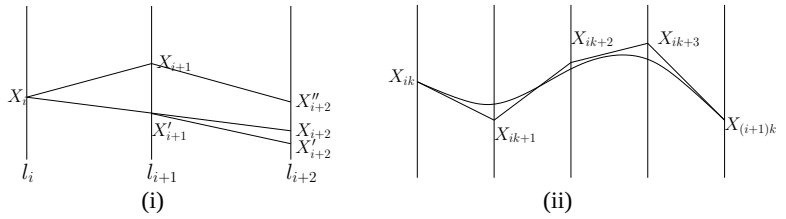


Fig. 2. (i) Speed restricted median; (ii) Smooth medians.

X_{i+1} is above X'_{i+1}. Assume the approximate median gets to the point X''_{i+2} at time t_{i+2}. Since the approximate median moves the fastest, the points above X_{i+1} will stay above X''_{i+2}. So only points between X_{i+1} and X'_{i+1}, whose number is at most $\delta n/3$, can cross $X_{i+1}X''_{i+2}$ without crossing $X'_{i+1}X_{i+2}$. Following the argument in Theorem 2, $X'_{i+1}X_{i+2}$ is crossed by at most $2\delta n/3$ lines. So $X_{i+1}X''_{i+2}$ is a δ-approximate median level. In addition, we can see that there are $n/2$ points below X'_{i+1} and X'_{i+2} respectively. So the number of points between X''_{i+2} and X'_{i+2} can only be less than the number of points between X_{i+1} and X'_{i+1}. So we can continue this scheme for the next time step. By the analysis above, we can get the following theorem.

Theorem 3. *The approximate median proposed above preserves all the properties of Theorem 2. In addition, the approximate median cannot move faster than the maximum speed of the n points.*

2.4 Smooth medians

For a set of moving points, the approximate median we computed above, follows a polygonal line composed of line segments. Although the approximate median moves

continuously, it may have to make sharp turns. In practice, a smooth trajectory is preferred. A curve is said to have C^k continuity if its k-th derivative is continuous.

Theorem 4. *Given an arrangement of n lines on the plane, we can compute a curve with C^k continuity which is a δ-approximate median level. The curve is determined by $O(k^2/\delta^2)$ control points and is computed in $O(k^2 n \log n/\delta^2)$ time.*

Proof. The idea is to use Bézier interpolation. Similarly we partition the plane into $4k^2/\delta^2$ strips, inside each there are at most $n^2\delta^2/(4k^2)$ intersections. We compute the medians X_i ($1 \leq i \leq 4k^2/\delta^2$) along the boundary of the strips. A curve c is computed as a degree k Bézier curve with X_i as control points. c has C^k continuity and passes through the control points X_{ik}, where $1 \leq i \leq 4k/\delta^2$ [22]. The part of c between points X_{ik} and $X_{(i+1)k}$ is completely determined by the control points in between, see Figure 2 (ii). From the variation diminishing property of the Bézier curves, if a line intersects the polygonal line defined by the control points (called control polygon) x times, then it intersects the Bézier curve at most x times. Since each line segment X_jX_{j+1} is crossed by at most $\delta n/k$ lines, the control polygon is crossed by at most δn times, so is the Bézier curve between X_{ik} and $X_{(i+1)k}$. This proves the theorem.

3 Pseudo kd-tree

3.1 Definition

Overmars [19] proposed the pseudo kd-tree as a dynamic data structure that admits efficient insertion and deletion. A *δ-pseudo kd-tree* is defined to be a binary tree created by alternately partitioning the points with vertical and horizontal lines, such that for each node with m points in the subtree, there are at most $(1/2 + \delta)m$ points in one subtree. A δ-pseudo kd-tree is an almost balanced tree with depth $\log_{2/(1-2\delta)} n$. We denote by $d(u)$ the depth of a node u. With the same analysis as the standard kd-tree, the range search query in a δ-pseudo kd-tree can be done in $O(n^{1/2+\varepsilon} + k)$ time, k is the number of points in the answer, $\varepsilon = \frac{1}{-2\log(1/2-\delta)} - 1/2$.

3.2 Maintaining the pseudo kd-tree

One way to maintain the pseudo kd-tree is to maintain the x-order and y-order sorted lists for all the points and update the tree when a point crosses the partition line. Maintaining the sorted lists will generate $\Omega(n^2)$ events. The subtree is rebuilt if the number of points in one child exceeds fraction $1/2 + \delta$. Each event has an amortized update cost of $O(\log n)$. Or in another way, we can make use of the dynamic data structure proposed by Overmars [19]. However, we have to maintain a forest of kd-trees instead of a single tree. Next we will show how to maintain a single kd-tree with only polylogarithmic update cost per event without any rebuilding. In the following part of this section, we assume the points move with constant velocity.

To maintain a pseudo kd-tree, we need to maintain a hierarchical partition of the points which always supports a δ-pseudo kd-tree. Since points are inserted into or deleted from a subtree, the trajectories of the points stored in a node are actually line

segments. Maintaining the partition line of a node involves finding the approximate median of a set of line segments. We define a *δ-approximate partition line* of depth k to be a δ-approximate median level in the arrangement of the trajectories of points stored in the subtree of a node with depth k. Using Theorem 1, we obtain the following.

Lemma 2. *A set of δ-approximate partition lines with total complexity $O(\frac{k+1}{\delta^2\alpha^{2k}})$ for all the nodes in the δ-pseudo kd-tree with depth k can be computed in $O(\frac{(k+1)n\log n}{\delta^2\alpha^{2k}})$ time, where $\alpha = 1/2 - \delta$.*

Proof. A key observation is that the combination of the trajectories of all the points in nodes of depth k is the arrangement of n straight lines. So the total number of intersections of the segments in nodes of depth k is bounded by $O(n^2)$. Assume a node of depth k has n_k points associated with it, $n_k \geq n\alpha^k$, $\alpha = 1/2 - \delta$. Denote by s_k the total number of line segments in the nodes of depth k. Denote by c_k the complexity of the δ-approximate partition lines for all the nodes of depth k. From Theorem 1, we have that $c_k = O(\frac{n^2}{\delta^2 n_k^2}) + O(\frac{s_k}{\delta n_k})$. When a point crosses the partition line at a node of depth k, it ends the trajectory in the old child and begins a trajectory in the new child. Therefore we have $s_{k+1} = 2c_k \times \delta n_k = 2O(\frac{n^2}{\delta n_k}) + 2s_k$. By induction we get $s_k = O(\frac{kn}{\delta\alpha^k})$, $c_k = O(\frac{k+1}{\delta^2\alpha^{2k}})$. The running time is bounded in the same way as in Theorem 1.

An event happens when a point crosses a partition line. Updating the pseudo tree involves moving a point from one subtree to the other, which costs $\log_{2/(1-2\delta)} n$. The number of events of depth k is bounded by $s_{k+1}/2$. So the total number of events is $O((n^2/\delta)\log_{2/(1-2\delta)} n)$. Computing the approximate hierarchy would cost $O(n^2)$ since that is the cost of computing the arrangement of n lines. In summary, we have

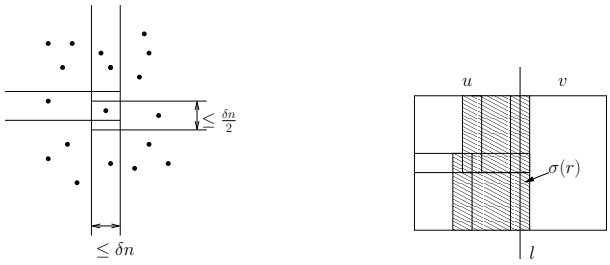
Theorem 5. *For a set of n moving points on the plane, we can find a moving hierarchical partition of the plane in $O(n^2)$ time which supports a δ-pseudo kd-tree at any time. The number of events is $O((n^2/\delta)\log_{2/(1-2\delta)} n)$ in total. Update cost for each event is $O(\log_{2/(1-2\delta)} n)$.*

For online maintenance, the ideas in 2.2 work for nodes of depth k as well. The difference is, points may come in and out of the range of a node. So inside one strip the number of intersections, endpoints of the line segments and flight plan changes should be bounded all at the same time. The endpoints of the line segments can be treated in the same way as the events of motion plan changes we described before. We omit the details here.

4 Overlapping kd-tree

4.1 Definition

In the second variant of the kd-tree, the bounding boxes associated with the children of a node can overlap. Assume $S(v)$ is the set of points stored in the subtree of a node v, $B(v)$ is the minimum bounding box of $S(v)$. Define $\sigma(w) = B(u) \cap B(v)$ to be the *overlapping region* of two children u and v of node w. A *δ-overlapping kd-tree* is a



(i) (ii)

Fig. 3. (i) δ-overlapping kd-tree; (ii) Only 2 grandchildren of u intersect line l.

perfectly balanced tree so that for each node w with m points in the subtree, there can be at most δm points in the overlapping region $\sigma(w)$, $0 < \delta < 1$. Figure 3 (i) shows the top two levels. The overlapping kd-tree has linear size and depth $O(\log n)$. ($\log$ refers to $\log_2$, if not specified.) We define a node to be x-partitioned (y-partitioned) if it is partitioned by a vertical (horizontal) line. Rectangular range-search query can be answered in the similar way as in the standard kd-tree.

Lemma 3. *Let l be a vertical (or horizontal) line, and let v, w be two x-partitioned (y-partitioned) nodes of a δ-overlapping kd-tree such that w is a descendant of v. If l intersects both $\sigma(v)$ and $\sigma(w)$, then $d(w) \geq d(v) + \log\gamma$, where $\gamma = \frac{1-2\delta}{2\delta}$.*

Theorem 6. *The range search query in a δ-overlapping kd-tree can be answered in $O(n^{1/2+\varepsilon} + k)$ time, where $\varepsilon = \log_\gamma 2$, $\gamma = \frac{1-2\delta}{2\delta}$, k is the number of points reported.*

Proof. Like the standard kd-tree, the only non-trivial part is to bound the number of nodes intersected by a vertical line l. Assume the point set is cut by a vertical line at the top level. l intersects the root r. Only when l intersects the overlapping region do we need to traverse both children u and v. However, if δ is not too large, of the four grandchildren of node u, at most two of them intersect line l, see Figure 3 (ii). This property remains true when we go down the tree until we meet a x-partitioned node w such that l intersects $\sigma(w)$. From Lemma 3, node w has depth at least $h = \log\gamma$. We then bound the number of intersected nodes from root to depth h. Notice that the number of intersected nodes of depth $2i+1$ is the same with that of depth $2i$, and they are both bounded by 2^{i+1}, for $1 \leq i \leq h/2 - 1$. The total number of intersected nodes from depth 1 to depth h is no more than $2 \times \sum_{i=0}^{j} 2^{i+1} = 2^{j+3} = 8\sqrt{\gamma}$. Assume the total number of nodes intersected by l is $C(n,\delta)$. We have $C(n,\delta) \leq 2\sqrt{\gamma}C(n/\gamma,\delta) + 8\sqrt{\gamma}$. By induction, $C(n,\delta) = O(n^{1/2+\varepsilon})$, where $\varepsilon = \log_\gamma 2$.

4.2 Maintaining the overlapping kd-tree

Maintaining the δ-overlapping kd-tree mainly involves tracking the number of points in the overlapping region. Consider the root of the kd-tree, assume n points are divided into $n/2$ red points and $n/2$ blue points by a vertical line at the beginning. W.l.o.g,

assume red points have bigger coordinates. Consider the time-space plane, the goal is to track the depth of the lower envelope of red lines in the arrangement of blue lines, similarly the depth of the upper envelope of blue lines in the arrangement of red lines. An event happens when those two numbers add up to δn. So at least $\delta n/2$ points in the overlapping region have the same color, suppose they are red. Then the highest blue point must be above $\delta n/2$ red points. We define a red-blue inversion to be the intersection of a red trajectory and a blue trajectory. So there must be at least $\Omega(\delta n)$ red-blue inversions since the last recoloring. The total recoloring events in the root of the kd-tree is bounded by $O(n/\delta)$, if the points follow pseudo-algebraic motion.

Since the combination of the trajectories of points in all nodes of depth k becomes the arrangement of the trajectories of n points, we can prove that the total recoloring events in the nodes of depth k of the δ-overlapping kd-tree is bounded by $O(n2^k/\delta)$. Therefore the total number of events is bounded by $O(n^2/\delta)$. When an event happens at a node u with m points stored in the subtree, we first recolor the δm points so that red points and blue points do not overlap, which costs δm. This may invoke the recoloring events at the children of u. So the total update cost is no more than $O(m \log m)$. A recoloring event at depth k costs $O(n \log n/2^k)$. So for all the recoloring events, the total update cost sum up to $O(n^2 \log n/\delta)$. The amortized cost for one event is therefore $O(\log n)$. Putting everything together, we obtain the following.

Theorem 7. *The total number of recoloring events for a δ-overlapping kd-tree for n points with pseudo-algebraic motion is $O(n^2/\delta)$. Each event has an amortized cost of $O(\log n)$.*

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