An Improved Lower Bound on the Growth Constant of Polyiamonds*

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Abstract. A polyiamond is an edge-connected set of cells on the triangular lattice. In this paper we provide an improved lower bound on the asymptotic growth constant of polyiamonds, proving that it is at least 2.8424. The proof of the new bound is based on a concatenation argument and on elementary calculus. We also suggest a nontrivial extension of this method for improving the bound further. However, the proposed extension is based on an unproven (yet very reasonable) assumption.

Keywords: Polyiamonds, lattice animals, growth constant.

1 Introduction

A *polyomino* of size n is an edge-connected set of n cells on the square lattice $\mathbb{Z}^2$. Similarly, a *polyiamond* of size n is an edge-connected set of n cells on the two-dimensional triangular lattice. *Fixed* polyiamonds are considered distinct if they have different *shapes* or *orientations*. In this paper we consider only fixed polyiamonds, and so we refer to them in the sequel simply as "polyiamonds." Figure 1 shows polyiamonds of size up to 5.

In general, a connected set of cells on a lattice is called a *lattice animal*. The fundamental combinatorial problem concerning lattice animals is "How many animals with n cells are there?" The study of lattice animals began in parallel more than half a century ago in two different communities. In statistical physics, Temperley [21] investigated the mechanics of macro-molecules, and Broadbent and Hammersley [7] studied percolation processes. In mathematics, Harary [12] composed a list of unsolved problems in the enumeration of graphs, and Eden [8] analyzed cell growth problems. Since then, counting animals has attracted much attention in the literature. However, despite serious efforts over the last 50 years, counting polyominoes is still far from being solved, and is considered [2] one of the long-standing open problems in combinatorial geometry.

The symbol $A(n)$ usually denotes the number of polyominoes of size n; See sequence A001168 in the On-line Encyclopedia of Integer Sequences (OEIS) [1].

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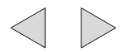
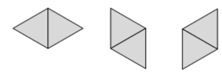
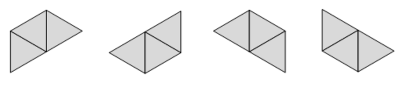


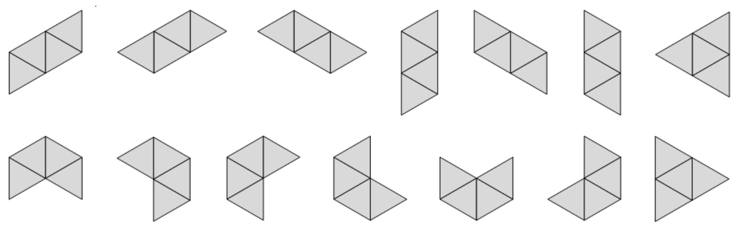
(a) $n = 1$: Two moniamonds



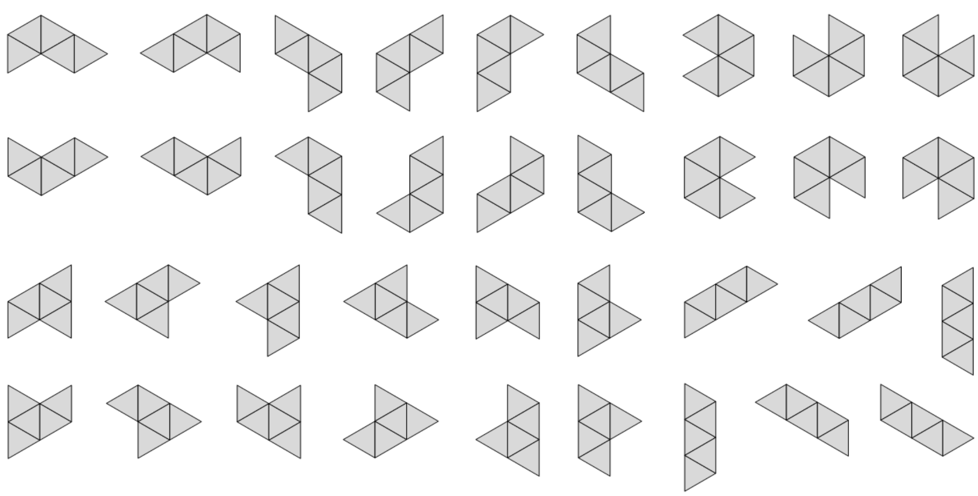
(b) $n = 2$: Three diamonds



(c) $n = 3$: 6 triamonds



(d) $n = 4$: 14 tetriamonds



(e) $n = 5$: 36 pentiamonds

Fig. 1. Polyiamonds of sizes $1 \leq n \leq 5$

Since no analytic formula for the number of animals is yet known for any nontrivial lattice, a great portion of the research has so far focused on efficient algorithms for *counting* animals on lattices, primarily on the square lattice. Elements of the sequence $A(n)$ are currently known up to $n = 56$ [13]. The growth constant of polyominoes was also treated extensively in the literature, and a few asymptotic results are known. Klarner [14] showed that the limit $\lambda := \lim_{n\to\infty} \sqrt[n]{A(n)}$ exists, and the main problem so far has been to evaluate this constant. The convergence of $A(n+1)/A(n)$ to λ (as $n\to\infty$) was proven only three decades later by Madras [17], using a novel pattern-frequency argument. The best-known lower and upper bounds on λ are 4.0025 [5] and 4.6496 [15], respectively. It is widely believed (see, e.g., [9, 10]) that $\lambda \approx 4.06$, and the currently best estimate, $\lambda = 4.0625696 \pm 0.0000005$, is due to Jensen [13].

In the same manner, let $T(n)$ denote the number of polyiamonds of size n (sequence A001420 in the OEIS). Elements of the sequence $T(n)$ were computed up to $n = 75$ [11, p. 479] using a transfer-matrix algorithm by Jensen [ibid., p. 173], adapting his original polyomino-counting algorithm [13]. Earlier counts were given by Lunnon [16] up to size 16, by Sykes and Glen [20] up to size 22,[1] and by Aleksandrowicz and Barequet [3] (extending Redelmeier's polyomino-counting algorithm [19]) up to size 31.

Similarly to polyominoes, the limits $\lim_{n\to\infty} \sqrt[n]{T(n)}$ and $\lim_{n\to\infty} T(n+1)/T(n)$ exist and are equal. Let, then, $\lambda_T = \lim_{n\to\infty} \sqrt[n]{T(n)}$ denote the growth constant of polyiamonds. Klarner [14, p. 857] showed that $\lambda_T \geq 2.13$ by taking the square root of 4.54, a lower bound he computed for the growth constant of animals on the rhomboidal lattice, using the fact that a rhombus is made of two neighboring equilateral triangles. This bound is also mentioned by Lunnon [16, p. 98]. Rands and Welsh [18] used renewal sequences in order to show that

$$\lambda_T \geq (T(n)/(2(1+\lambda_T)))^{1/n} \tag{1}$$

for any $n \in \mathbb{N}$. Substituting the easy upper bound[2] $\lambda_T \leq 4$ in the right-hand side of this relation, and knowing at that time elements of the sequence $T(n)$ for $1 \leq n \leq 20$ only (data provided by Sykes and Glen [20]), they used $T(20) =$

[1] Note that in this reference the lattice is called "honeycomb" (hexagonal) and the terms should be doubled. The reason for this is that the authors actually count clusters of *vertices* on the hexagonal lattice, whose connectivity is the same as that of *cells* on the triangular lattice, with no distinction between the two possible orientations of the latter cells. This is why polyiamonds are often regarded in the literature as site animals on the hexagonal lattice, and polyhexes (cell animals on the hexagonal lattice) are regarded as site animals on the triangular lattice, which sometimes causes confusion.

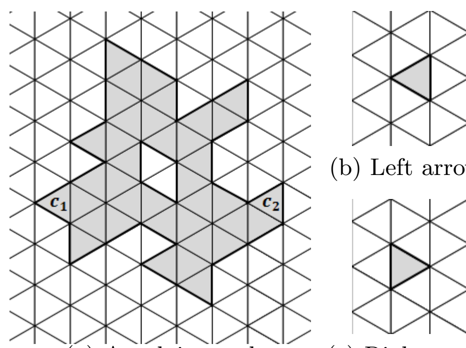
[2] This easy upper bound, based on an idea of Eden [8] was described by Lunnon [16, p. 98]. Every polyiamond P can be built according to a set of $n-1$ "instructions" taken from a superset of size $2(n-1)$. Each instruction tells us how to choose a lattice cell c, neighboring a cell already in P, and add c to P. (Some of these instruction sets are illegal, and some other sets produce the same polyiamonds, but this only helps.) Hence, $\lambda_T \leq \lim_{n\to\infty} \binom{2(n-1)}{n-1}^{1/n} = 4$.

173, 338, 962 to show that $\lambda_T \geq (T(20)/10)^{1/20} \approx 2.3011$.[3] Nowadays, since we know $T(n)$ up to $n = 75$,[4] we can obtain, using the same method, that $\lambda_T \geq (T(75)/10)^{1/75} \approx 2.7714$. We can even do slightly better than that. Substituting in Equation (1) the upper bound $\lambda_T \leq 3.6050$ we obtained elsewhere [6], we see that $\lambda_T \geq (T(75)/(2(1 + 3.6050)))^{1/75} \approx 2.7744$. However, we can still improve on this.

Based on existing data, it is believed [20] (but has never been proven) that $\lambda_T = 3.04 \pm 0.02$. In this paper we improve the lower bound on λ_T, showing that $\lambda_T \geq 2.8424$. The new lower bound is obtained by combining a concatenation argument with elementary calculus.

2 Preliminaries

We orient the triangular lattice as is shown in Figure 2(a), and define a lexico-



(a) A polyiamond (b) Left arrow (c) Right arrow

Fig. 2. Polyiamonds on the triangular lattice

graphic order on the cells of the lattice as follows: A triangle t_1 is *smaller* than triangle $t_2 \neq t_1$ if the lattice column of t_1 is to the left of the column of t_2, or if t_1, t_2 are in the same column and t_1 lies below t_2. Triangles that look like a “left arrow” (Figure 2(b)) are of Type 1, and triangles that look like a “right arrow” (Figure 2(c)) are of Type 2. In addition, let $T_1(n)$ be the number of polyiamonds of size n whose *largest* (top-right) triangle is of Type 1, and let $T_2(n)$ be the number of polyiamonds of size n whose *largest* triangle is of Type 2.

Let $x(n)$ $(0 < x(n) < 1)$ denote the fraction of polyiamonds of Type 1 out of all polyiamonds of size n, that is, $T_1(n) = x(n)T(n)$ and $T_2(n) = (1-x(n))T(n)$. In addition, let $y(n) = T_2(n)/T_1(n) = (1 - x(n))/x(n)$ denote the ratio between polyiamonds of Type 2 and polyiamonds of Type 1 (all of size n).

[3] We wonder why Rands and Welsh did not use $T(22) = 1,456,891,547$ (which was also available in their reference [20]) to show that $\lambda_T \geq (T(22)/10)^{1/22} \approx 2.3500$.

[4] $T(75) = 15,936,363,137,225,733,301,433,441,827,683,823$.

A concatenation of two polyiamonds P_1, P_2 is the union of P_1 and a translated copy of P_2, so that the largest triangle of P_1 is attached to the smallest triangle of P_2, and all cells of P_1 are smaller than the translates of cells of P_2. We use a concatenation argument in order to improve the lower bound on λ_T.

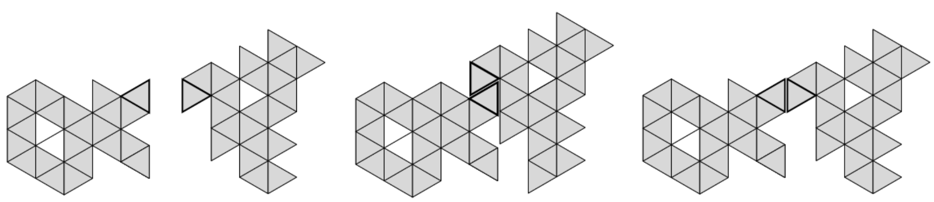
3 The Bound

Our proof of a lower bound on λ_T uses the division of polyiamonds into Type 1 and Type 2, but does not employ the asymptotic proportion between the two types.

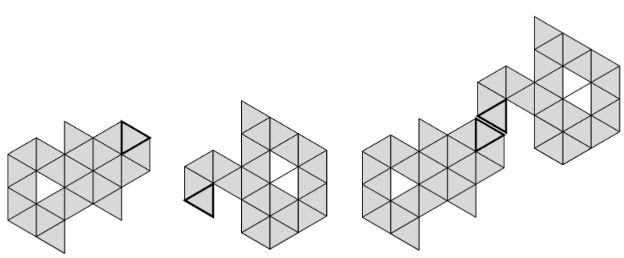
Theorem 1. $\lambda_T \geq 2.8424$.

Proof. First note that by rotational symmetry, the number of polyiamonds of size n, whose *smallest* (bottom-left) triangle is of Type 2, is $T_1(n)$. Similarly, the number of polyiamonds, whose *smallest* triangle is of Type 1, is $T_2(n)$.

We proceed with a concatenation argument tailored to the specific case of the triangular lattice. Interestingly, not all pairs of polyiamonds of size n can be concatenated. In addition, there exist many polyiamonds of size $2n$ which cannot be represented as the concatenation of two polyiamonds of size n. Let us count carefully the amount of pairs of polyiamonds that can be concatenated.



(a) Two polyiamonds (b) Vertical concatenation (c) Horizontal concatenation
Two attachments of Type-1 and Type-2 triangles



(d) Two polyiamonds (e) Vertical concatenation
A single attachment of Type-2 and Type-1 triangles

Fig. 3. Possible concatenations of polyiamonds

- Polyiamonds, whose largest triangle is of Type 1, can be concatenated only to polyiamonds whose smallest triangle is of Type 2, and this can be done in two different ways (see Figures 3(a–c)). There are $2(T_1(n))^2$ concatenations of this kind.
- Polyiamonds, whose largest triangle is of Type 2, can be concatenated, in a single way, only to polyiamonds whose smallest triangle is of Type 1 (see Figures 3(d,e)). There are $(T_2(n))^2$ concatenations of this kind.

Altogether, we have $2(T_1(n))^2+(T_2(n))^2$ possible concatenations, and, as argued above,

$$2(T_1(n))^2 + (T_2(n))^2 \leq T(2n). \tag{2}$$

Let us now find an efficient lower bound on the number of concatenations. Equation (2) can be rewritten as $T(2n) \geq 2(x(n)T(n))^2 + ((1-x(n))T(n))^2 = (3x^2(n) - 2x(n) + 1)T^2(n)$. Elementary calculus shows that the function $f(x) = 3x^2 - 2x + 1$ assumes its minimum at $x = 1/3$ and that $f(1/3) = 2/3$. Hence,

$$\frac{2}{3}T^2(n) \leq T(2n).$$

By simple manipulations of this relation, we obtain that

$$\big(\frac{2}{3}T(n)\big)^{1/n} \leq \big(\frac{2}{3}T(2n)\big)^{1/(2n)}.$$

This implies that the sequence $\big(\frac{2}{3}T(k)\big)^{1/k}, \big(\frac{2}{3}T(2k)\big)^{1/(2k)}, \big(\frac{2}{3}T(4k)\big)^{1/(4k)}, \ldots$ is monotone increasing for any value of k, and, as a subsequence of $\Big(\big(\frac{2}{3}T(n)\big)^{1/n}\Big)$, it converges to λ_T too. Therefore, any term of the form $\big(\frac{2}{3}T(n)\big)^{1/n}$ is a lower bound on λ_T. In particular, $\lambda_T \geq (\frac{2}{3}T(75))^{1/75} \approx 2.8424$. □

4 Convergence, Interval Containment, and Monotonicity

First, we show a simple relation between the number of polyiamonds of Type 2 of size n and the number of polyiamonds of Type 1 of size n+1,

Observation 2 $T_2(n) = T_1(n+1)$ *(for all* $n \in \mathbb{N}$*).*

This simple observation follows from the fact that if the largest triangle t of a polyiamond P is of Type 1, then the only possible neighboring triangle of t within P is the triangle immediately below it, and so, removing t from P will not break P into two parts. This implies a bijection between Type-1 polyiamonds of size n+1 and Type-2 polyiamonds of size n.

An immediate consequence is that $y(n) = T_1(n+1)/T_1(n)$, a fact which we will use later frequently (see, e.g., the proof of Theorem 4(ii) below).

Next, we prove the convergence of the sequences $(T_1(n+1)/T_1(n))$, $x(n)$, and $y(n)$, and find their limits.

Observation 3 *The sequence* $(T_1(n+1)/T_1(n))_{n=1}^{\infty}$ *converges.*

Indeed, polyiamonds of Type 1 fulfill all the premises of Madras's Ratio Limit Theorem [17, Thm. 2.2]. (A more detailed explanation is given in the full version of the paper.) Hence, the limit $\lim_{n\to\infty} T_1(n+1)/T_1(n)$ exists and is equal to some constant μ, whose value is specified (indirectly) in the following theorem.

Theorem 4. *(i)* $\mu = \lambda_T$*; (ii)* $\lim\limits_{n\to\infty} y(n) = \lambda_T$*; (iii)* $\lim\limits_{n\to\infty} x(n) = 1/(\lambda_T + 1)$.

Proof.

(i) On the one hand, by Madras [17] we know that the limit $\lambda_T := \lim_{n\to\infty} T(n+1)/T(n)$ exists. On the other hand, we have that

$$\frac{T(n+1)}{T(n)} = \frac{T_1(n+1)+T_2(n+1)}{T_1(n)+T_2(n)} = \frac{T_1(n+1)+T_1(n+2)}{T_1(n)+T_1(n+1)} = \frac{1+\frac{T_1(n+2)}{T_1(n+1)}}{\frac{T_1(n)}{T_1(n+1)}+1} \underset{n\to\infty}{\longrightarrow} \frac{1+\mu}{\frac{1}{\mu}+1} = \mu.$$

(Note that the convergence in the last step relies on the fact that the sequence $(T_1(n+1)/T_1(n))$ has a limit μ.) Hence, $\mu = \lambda_T$.

(ii)
$$y(n) = \frac{T_2(n)}{T_1(n)} = \frac{T_1(n+1)}{T_1(n)} \underset{n\to\infty}{\longrightarrow} \mu = \lambda_T.$$

(iii)
$$x(n) = \frac{T_1(n)}{T(n)} = \frac{T_1(n)}{T_1(n)+y(n)T_1(n)} = \frac{1}{1+y(n)} \underset{n\to\infty}{\longrightarrow} \frac{1}{\lambda_T+1}.$$ □

Now, we show relations between bounds on the entire sequence $(x(n))$ and bounds on the entire sequence $(y(n)) \equiv (T(n+1)/T(n))$.

Theorem 5. *Let* d, e *be two constants.*

(i) $x(n) \geq 1/d$ *for all* $n \in \mathbb{N} \longleftrightarrow T(n+1)/T(n) \leq d-1$ *for all* $n \in \mathbb{N}$*;*

(ii) $x(n) \leq 1/e$ *for all* $n \in \mathbb{N} \longleftrightarrow T(n+1)/T(n) \geq e-1$ *for all* $n \in \mathbb{N}$.

Proof.

(i) First,

$$x(n) \geq \frac{1}{d} \longleftrightarrow \frac{T_1(n)}{T(n)} \geq \frac{1}{d} \longleftrightarrow \frac{T_1(n)}{T_1(n)+T_1(n+1)} \geq \frac{1}{d} \longleftrightarrow \frac{T_1(n)+T_1(n+1)}{T_1(n)} \leq d \longleftrightarrow 1+\frac{T_1(n+1)}{T_1(n)} \leq d \longleftrightarrow \frac{T_1(n+1)}{T_1(n)} \leq d-1.$$

Second,

$$\frac{T(n+1)}{T(n)} = \frac{T_1(n+1)+T_1(n+2)}{T_1(n)+T_1(n+1)} = \frac{1+T_1(n+2)/T_1(n+1)}{T_1(n)/T_1(n+1)+1} \leq \frac{1+(d-1)}{1/(d-1)+1} = d-1.$$

(ii) The proof is completely analogous to that of item (i). □

Finally, we observe the relation between the directions of monotonicity (if exist) of the sequences $(x(n))$ and $(y(n))$.

Observation 6 *If any of the two sequences* $(x(n))$ *and* $(y(n))$ *is monotone, then the other sequence is also monotone but in the opposite direction.*

Indeed, this follows immediately from the equality $y(n) = (1 - x(n))/x(n) = 1/x(n) - 1$. The available data suggest that $(x(n))$ be monotone decreasing and that $(y(n))$ be monotone increasing. Note that the monotonicity of $x(n)$ (or $y(n)$) neither implies, nor is implied by, the monotonicity of $(T(n+1)/T(n))$.

5 Conditional Lower Bounds

Let $L(n)$ denote the number of animals on a lattice $\mathcal{L}$. It is widely believed (but has never been proven) that asymptotically

$$L(n) \sim C_{\mathcal{L}} n^{-\theta_{\mathcal{L}}} \lambda_{\mathcal{L}}, \tag{3}$$

where $C_{\mathcal{L}}$, $\theta_{\mathcal{L}}$, and $\lambda_{\mathcal{L}}$ are constants which depend on $\mathcal{L}$.[5] If this were true, then it would guarantee the existence of the limit $\lambda_L := \lim_{n\to\infty} L(n+1)/L(n)$, the growth constant of animals on the lattice $\mathcal{L}$. As was mentioned in the introduction, the existence of this limit was proven by Madras [17] without assuming the relation in Equation (3). Furthermore, it is widely believed that the "ratio sequence" $(L(n+1)/L(n))$ is *monotone increasing*.[6] Available data (numbers of animals on various lattices) support this belief, but it was unfortunately never proven. Such monotonicity of the ratio sequence would imply that the entire sequence lies below its limit. (In fact, this may be the case even without monotonicity.) Consequently, every element of the ratio sequence would be a lower bound on the growth constant of animals on $\mathcal{L}$. In particular, it would imply the lower bound $\lambda_T \geq T(75)/(74) \approx 2.9959$, only 0.05 short of the estimated value.

Another plausible direction for setting a lower bound on λ_T would be to prove that the entire sequence $(T(n+1)/T(n))$ lies below some constant $c > \lambda_T$, that is, $T(n+1)/T(n) \leq c$ for all $n \in \mathbb{N}$. As we demonstrate at the end of this section, proving this relation for *any* arbitrary constant c, even a very large one, would improve the lower bound on λ_T, provided in Section 3.

In the proof of Theorem 1, we considered the three possible kinds of concatenations of the largest and the smallest triangles in two polyiamonds of size n. In

[5] In fact, it is widely believed (but not proven) that the constant θ is common to *all* lattices in the same dimension. In particular, there is evidence that $\theta = 1$ for all lattices in two dimensions.

[6] Madras [17, Prop. 4.2] proved "almost monotonicity" for all lattices, namely, that $L(n+2)/L(n) \geq (L(n+1)/L(n))^2 - \Gamma_{\mathcal{L}}/n$ for all sufficiently large values of n, where $\Gamma_{\mathcal{L}}$ is a constant which depends on $\mathcal{L}$. Note that if $\Gamma_{\mathcal{L}} = 0$, then we have $L(n+2)/L(n+1) \geq L(n+1)/L(n)$, i.e., that the ratio sequence of $\mathcal{L}$ is monotone increasing.

fact, more compositions which preserve the lexicographical order of the polyiamonds can be obtained by using horizontal attachments of Type-1 and Type-2 triangles which are not the largest or smallest triangles in the respective polyiamonds. The following cases, shown in Figure 4, are categorized by the type of extreme triangles (t_1, t_2) in a pair of composed polyiamonds, where t_1 is the largest triangle of the smaller polyiamond and t_2 is the smallest triangle of the larger polyiamond. Therefore, these cases are distinct. In the figure, attached triangles are colored red, the direction of the attachment is marked by a small arrow, and an "×" sign marks an area free of triangles of the polyiamonds. The triangles in gray are the largest (or smallest) in polyiamonds of size less than n, which are then extended to full size-n polyiamonds. (In Case (c), the red and gray triangles identify.) For the purpose of further computation, let x_i denote the ratio $T_1(n-i)/T(n-i)$, for $i = 0, 1, 2, 3$, and let c be an upper bound on the sequence $(T(n+1)/T(n))$. All compositions below do not appear in the proof of Theorem 1.

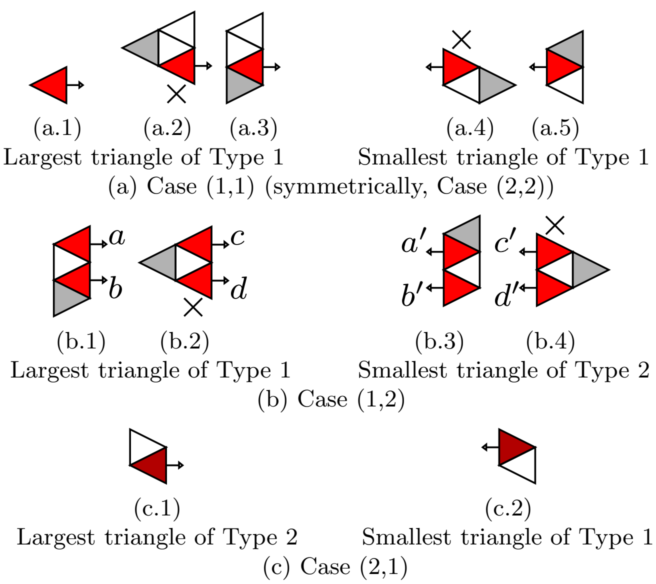


Fig. 4. Additional compositions of polyiamonds

(1,1) In this type of composition, each one of the polyiamond types shown in Figures 4(a.1–a.3) is composed with each one of the types shown in Figures 4(a.4,a.5). Specifically, the attached triangle in types (a.1) and (a.2,a.3) is the largest and third largest, respectively, and the attached triangle in types (a.4,a.5) is the second largest. The number of polyiamonds whose largest triangle is of Type 1 (Figure 4(a.1)) is $T_1(n)$, and among these polyiamonds, the number of polyiamonds shown in Figure 4(a.2) is $T_1(n-3)$.

One way to see it is that, for every polyiamond of of size $n-3$ whose largest triangle (shown in gray) is of Type 1, a column-like polyiamond composed of three triangles whose largest is of Type 1 is attached to the right of it. With a similar reasoning, there are $T_2(n-3)$, $T_1(n-2)$, and $T_2(n-2)$ polyiamonds of the types shown in Figures 4(a.3–a.5), respectively. Hence, the number of compositions in this category is

$$\big(T_1(n) + T_1(n-3) + T_2(n-3)\big)\big(T_1(n-2) + T_2(n-2)\big).$$

By observation 2, we can substitute $T_2(n-i)$ by $T_1(n-i+1)$, and by the definition of x_i, replace $T_1(n-i)$ by $x_i T(n-i)$. Then, by the definition of c, we have $T(n-i) \geq \frac{T(n)}{c^i}$, and we conclude that in this category we have

$$\begin{aligned}\big(x_0T(n) + x_3T(n-3) + x_2T(n-2)\big)\big(x_2T(n-2) + x_1T(n-1)\big)\\ \geq \left(x_0 + \frac{x_2}{c^2} + \frac{x_3}{c^3}\right)\left(\frac{x_1}{c} + \frac{x_2}{c^2}\right)T^2(n). \quad (4)\end{aligned}$$

Notice that, in principal, the attachment of smaller and larger triangles (of the two polyiamonds, respectively) can be counted as well. However, in such cases, higher orders of $1/c$ would appear in Equation 4. The contribution of such compositions is relatively insignificant, thus, they are not considered.

(1,2) In this case, we count attachments of the largest and the third largest triangles in the smaller polyiamonds, and the smallest and third smallest triangles in the larger polyiamonds; see Figures 4(b.1–b.4). Table 1 lists all possible

Attached Triangles	Number of Compositions
(a ↔ a',c')	$T_2^2(n-3) + T_1(n-3)T_2(n-3)$
(b ↔ a',b',c',d')	$2T_2^2(n-3) + 2T_1(n-3)T_2(n-3)$
(c ↔ a',c')	$T_1^2(n-3) + T_1(n-3)T_2(n-3)$
(d ↔ a',b',c',d')	$2T_1^2(n-3) + 2T_1(n-3)T_2(n-3)$

Table 1. Additional compositions in Case (1,2)

attachments and their corresponding numbers of compositions. To sum up, the total number of compositions of this category is

$$3T_1^2(n-3) + 6T_1(n-3)T_2(n-3) + 3T_2^2(n-3) \geq 3\Big(\frac{x_2}{c^2} + \frac{x_3}{c^3}\Big)^2 T^2(n).$$

(2,1) The numbers of polyiamonds shown in Figures 4(c.1,c.2) are both $T_1(n-1)$. Therefore, there are

$$T_1^2(n-1) \geq \frac{x_1^2}{c^2}T^2(n)$$

compositions in this category.

(2,2) By rotational symmetry, the number of compositions in this category is the same as that of Case (1,1).

Summing up the four cases and the three cases in the proof of Theorem 1, we obtain the relation

$$\left(1 - 2x_0 + 3{x_0}^2 + 2\left(x_0 + \frac{x_2}{c^2} + \frac{x_3}{c^3}\right)\left(\frac{x_1}{c} + \frac{x_2}{c^2}\right) + \frac{x_1^2}{c^2} + 3\left(\frac{x_2}{c^2} + \frac{x_3}{c^3}\right)^2\right) T^2(n) \\ \leq T(2n) \qquad (5)$$

In order to set a good (high) lower bound on λ_T, we need good lower bounds on x_0, x_1, x_2, x_3 and a good upper bound on c. Then we obtain a relation of the form $\mu \cdot T^2(n) \leq T(2n)$, where μ is a constant, and proceed in the same way as in the proof of Theorem 1. If we denote the 4-variable polynomial multiplying $T^2(n)$ by $f(x_0, x_1, x_2, x_3)$, we find that $f(\cdot, \cdot, \cdot, \cdot)$ is monotone decreasing when x_1, x_2, x_3 decrease, so its infimum is obtained when $x_1 = x_2 = x_3 = 0$, which is useless.

However, the actual data show that the sequence $(T(n{+}1)/T(n))$ is monotone increasing with the limit $\lambda_T \approx 3.04$. Assume, then, *without proof* that $(T(n+1)/T(n)) \leq 4$ for all $n \in \mathbb{N}$. By Theorem 5, this implies that $x(n) > 0.2$ for all values of n. Using these bounds, we see that the left-hand side of Equation (5) becomes a quadratic function $g(x_0) = 3x_0^2 - 1.875x_0 + 1.00519$. (All computations were done with a much higher precision.) Elementary calculus shows that $g(x_0)$ assumes its minimum at $x_0 = 0.3125$ and that $g(0.3125) = 0.7122$. Hence,

$$0.7122 \cdot T^2(n) \leq T(2n).$$

With manipulations similar to those in the proof of Theorem 1, it can be shown that the sequence $\left(\left(0.7122 \cdot T(2^i k)\right)^{1/(2^i k)}\right)_{i=0}^{\infty}$ is monotone increasing for any value of k, and, as a subsequence of $\left(\left(0.7122 \cdot T(n)\right)^{1/n}\right)$, it converges to λ_T as well. Therefore, any term of the form $\left(0.7122 \cdot T(n)\right)^{1/n}$ is a lower bound on λ_T. In particular, $\lambda_T \geq (0.7122 \cdot T(75))^{1/75} \approx 2.8449$, which is a slight improvement over the lower bound on λ_T, shown in Section 3, and is pending only the correctness of the assumption that $T(n+1)/T(n) \leq 4$ for all $n \in \mathbb{N}$.

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