

Collision Detection for Deforming Necklaces

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ABSTRACT

In this paper, we propose to study deformable necklaces — flexible chains of balls, called beads, in which only adjacent balls may intersect. Such objects can be used to model macro-molecules, muscles, rope, and other ‘linear’ objects in the physical world. In this paper, we exploit this linearity to develop geometric structures associated with necklaces that are useful in physical simulations. We show how these structures can be implemented efficiently and maintained under necklace deformation. In particular, we study a bounding volume hierarchy based on spheres built on a necklace. Such a hierarchy is easy to compute and is suitable for maintenance when the necklace deforms, as our theoretical and experimental results show. This hierarchy can be used for collision and self-collision detection. In particular, we achieve an upper bound of $O(n \log n)$ in two dimensions and $O(n^{2-2/d})$ in d -dimensions, $d \geq 3$, for collision checking. To our knowledge, this is the first sub-quadratic bound proved for a collision detection algorithm using predefined hierarchies. In addition, we show that the power diagram, with the help of some additional mechanisms, can be also used to detect self-collisions of a necklace in certain ways complementary to the sphere hierarchy.

Categories and Subject Descriptors

F.2.2 [Theory of Computation]: Analysis of Algorithms and Problem Complexity—*Nonnumerical Algorithms and Problems*; I.6.m [Computing Methodologies]: Simulation and Modeling—*Miscellaneous*

General Terms

Algorithms, Experimentation

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Keywords

deformable chains, collision detection, bounding volume hierarchy, power diagram

1. INTRODUCTION

The computational study of geometry has been motivated in part by the desire to model physical systems whose description naturally involves geometric attributes, such as shape. Indeed, efficient algorithms and data structures for handling geometric data are needed in almost every computational field that deals with the physical world, including computer graphics, computer vision, robotics, geographic information systems, spatial databases, molecular biology, and scientific computing. Recently there has been increased interest in modeling time-varying phenomena and this has naturally led to the study of geometric objects under motion.

Early work on moving objects in computational geometry, initiated by Atallah [3], focused on bounding the number of combinatorial changes in geometric structures as the objects move along prescribed trajectories [1]. In the late 1980s algorithms were developed for detecting and computing the actual changes in geometric structures as objects undergo motion, e.g. [23, 2]. However, these results assume an off-line model of motion in which the trajectories are given in advance, which is a relatively uncommon situation. More recently, the *Kinetic Data Structures Framework* (or KDS for short) [13, 4] was introduced to allow on-line changes to the motion. In the KDS framework certificates are used to ensure the correctness of a certain computation of the attribute of interest. Motion can cause certificates to fail; at such events the KDS repair mechanism is invoked to fix the attribute computation so that the simulation can proceed. Though the KDS view has provided an elegant framework for the analysis of motion algorithms, certain requirements limit its applicability, especially the need to predict certificate failure times. This is hard to do in many real-world situations where a physical system is evolving according to an ordinary or partial differential equation, since the motion plans of the simulation elements are not known in closed form, and time-stepping methods must be used.

Another aspect of motion that has not been adequately modeled in previous work is that objects in the world are often organized into groups and hierarchies and the motions of objects in the same group are highly correlated. For example, though not all points in an elastic bouncing ball follow exactly the same rigid motion, the trajectories of nearby points are very similar and the overall motion

is best described as the composition of a global rigid motion with a small local deformation. Similarly, the motion of an articulated figure, such as a man walking, is most succinctly described as a set of relative motions, say that of the upper right arm relative to the torso, rather than by giving the trajectory of each body part in world coordinates. All theoretical analysis to-date are based on the assumption of independently moving objects. By ignoring such motion coherence we run the danger of developing sub-optimal algorithms that do not exploit well the structure of the problem. A similar call for realistic input models in geometric algorithms was made in [7].

To begin addressing some of these issues, in this paper we propose to study a model for deformable ‘linear’ objects such as macromolecules, muscles, rope, etc. Though our objects live in 2-D or 3-D, they have an essential one-dimensional character that we heavily exploit in our algorithms. It is customary in engineering to model physical objects by breaking them up into ‘elements’ of various types — indeed this is the essence of the Finite-Element method [22]. Instead of a standard type of element (tetrahedra or hexahedra), we have chosen (potentially partially overlapping) spheres as our basic elements. The use of spheres simplifies substantially the basic geometric calculations and allows us to focus on the combinatorial issues that form our main interest. There is actually a literature on using spheres in engineering modeling [6] and for biomolecules spheres are obviously the right choice.

We call our linear objects *necklaces* and the spherical elements used to model them *beads*. The exact way in which a necklace moves and deforms depends on the physical model used and is application dependent. Our focus will be instead on tracking different geometric attributes of a necklace, such as its power diagram or a bounding sphere hierarchy that are useful in applications. For example, most forces in nature are short range and the power diagram provides useful proximity information among the beads. Collision and self-collision detection are important in most physical simulation settings and bounding volume hierarchies and the power diagram are both useful collision detection tools in such contexts. Since we do not model the physics, we take a *black box* model of the physical simulation. We assume that at certain times (the time steps of the simulation) an oracle moves the beads forming the necklace according to the underlying physics and reports their new position back to us. Though in general every single bead moves at every step, we assume that the time steps chosen by the simulator are such that at each step the motion of each bead is small when compared to the overall scale of the simulation. Unlike the kinetic data structure setting where we have explicit motion laws and can predict certificate failures, here we are called upon to repair a geometric structure after small displacements of its defining elements.

2. SUMMARY OF THE RESULTS

The two geometric tools we investigate in this paper are a bounding sphere hierarchy and the power diagram. The specific sphere hierarchy we use is based on bounding volumes which are the minimum containing spheres of the enclosed geometry. This gives each bounding sphere a small combinatorial description: such a sphere is defined by at most four of the underlying spherical elements. As our necklace deforms, both this sphere hierarchy and its power diagram consist of geometric elements (spheres, simplices) that also change continuously. But their combinatorial descriptions change at only discrete events. This event-based description update makes it possible to maintain them efficiently under motion. Both these structures can be used for proximity determination and collision checking. However, one of the key experimental results of this paper is that the sphere hierarchy is far more stable than the power diagram under deformation.

In general, a bounding volume hierarchy is formed by creating a balanced recursive partitioning of the underlying geometry and computing a bounding volume enclosing each group. Once the partitioning is determined, there are two ways to form the bounding volume hierarchy — one is to compute a tight bounding volume on the geometry in each group, which we call the ‘wrapped hierarchy’, and the other is to compute the bounding volume of the bounding volumes of the children subgroups, which we call the ‘layered hierarchy’. Clearly, the wrapped hierarchy is always tighter than the layered hierarchy. In this paper, we first study the relationship between the two hierarchies. We show the somewhat surprising result that, in the worst case, a bounding sphere in the layered hierarchy is at most a factor of $\sqrt{\log n}$ bigger than the corresponding one in the wrapped hierarchy, and this bound is tight. Furthermore, the bound holds in any dimension.

One important application of bounding hierarchies is in collision and self-collision checking. While such methods work well in practice, in the worst case nothing better than the trivial quadratic bound was previously known. We show that, with a simple heuristic, the folklore self-collision checking method using the sphere hierarchy achieves sub-quadratic time bounds: $O(n \log n)$ in two dimensions, and $O(n^{2-2/d})$ in d -dimensions for $d \geq 3$ — to our knowledge, this is the first sub-quadratic worst-case bound for collision detection algorithms using bounding volume hierarchies. The power diagram is another tool that people often use to deal with balls. While it has been known that the closest pair of a set of disjoint balls defines an edge in the power diagram ([15]), that result does not apply directly to our problem since we allow adjacent spherical elements to intersect. We show that the power diagram can be used to compute the closest pair in a deforming necklace as well. It is interesting to note that the worst-case for the sphere hierarchy occurs for highly packed necklaces, while these are actually very favorable cases for the power diagram — in such cases the power diagram size is linear in all dimensions. In [18], kinematic chains similar to our necklaces are studied, although with a different motivation. As in our paper, they also use bounding volume hierarchies for collision detection and achieve similar results on the complexity of collision checking.

In Section 3 we present the formal setting of our beads and necklaces. Section 4 discusses the precise bounding sphere hierarchy we have chosen to implement and the reasons for our choice. We also present verification and repair algorithms for maintaining the hierarchy as the necklace deforms, as well as compare the tightness of the wrapped and layered hierarchies. Section 5 presents a number of combinatorial results about collision detection using the sphere hierarchy or the power diagram. Section 6 discusses computational experiments for a number of different necklace scenarios, both static and dynamic, that study the performance of our sphere hierarchy and the power diagram under realistic conditions. Finally Section 7 concludes the paper.

3. BEADS AND NECKLACES

We give some definitions. A *necklace* consists of a sequence of n closed balls $\mathcal{B} = \{B_1, B_2, \dots, B_n\}$, called *beads*, in the Euclidean space E^d . We assume that only adjacent balls along the necklace may intersect and no ball is fully contained in another set of balls. In some contexts we make further assumptions. These include:

uniformity: there is a constant $\alpha \geq 1$ such that the ratio of the radii of any two balls in a necklace is in the interval $[1/\alpha, \alpha]$;

limited overlap: two consecutive balls B_i and B_{i+1} along a necklace are allowed to overlap, but the angle of their normals at a common point on their surfaces is bounded below by β .

Whatever conditions we adopt, we assume that they are maintained by the underlying physics causing a necklace to move or deform. We remark that similar ‘necklace conditions’ were studied for establishing the optimality of tours in the plane [8].

4. THE WRAPPED HIERARCHY

4.1 Definition and properties

Bounding volume hierarchies have been extensively used as a data structure for collision detection. To build such a hierarchy, a specific geometric shape is selected as the bounding volume of choice. Common choices are axis-aligned bounding boxes (AABBs), arbitrarily oriented bounding boxes (OBBs) [12], and spheres [21, 17]. The choice of a bounding shape usually presents a trade-off between the ease of testing for intersection two such shapes, and the total number of bounding volume checks required. Once the bounding volume shape is chosen, we can proceed to build a bounding volume hierarchy for a given object A . This can be done by several different methods. The object A or its surface is covered by a set of elementary bounding volumes and these are then aggregated and the object geometry enclosed by each aggregate is further covered by a bounding volume, and so on in a hierarchical fashion, until A has been entirely enclosed in a single bounding volume. Once hierarchies for rigid objects A and B have been computed, they can be used for repeated intersection testing among the objects A and B in different positions and poses. For given placements of A and B , their respective hierarchies are refined only to the coarsest level at which the primitive shapes in the two hierarchies can be shown to be pairwise disjoint — or until an intersection is detected.

Unquestionably, bounding volume hierarchies have been among the most successful methods used for collision detection in large and complex virtual environments and physical simulations. However, if the object bounded by a hierarchy not only moves but also deforms, the use of hierarchies becomes more problematic — current methods provide only a set of bad alternatives. One can expand the bounding volumes to allow some limited shape variability, but then more intersection tests will be needed because the bounding volumes do not fit as tightly. Or one can recompute the bounding volume hierarchy at each time step, but the cost of that is significant. One of the key contributions of this paper is to propose a bounding volume hierarchy for our deforming necklaces that can be efficiently maintained under deformation and remain tightly fitting.

The various hierarchies discussed above aggregate bounding volumes based on spatial proximity. Spatial proximity is variable over time and cannot be used as a reliable guide to aggregation especially when an object undergoes large deformations. We have decided to base the hierarchy we will use on *topological proximity* in the object, because this notion of proximity is preserved under deformation. For our linear necklace this gives us an especially simple rule for aggregation: we build a balanced binary tree on the sequence of beads $\{B_1, B_2, \dots, B_n\}$ so that the intermediate aggregates correspond to the sets of leaves that are descendants of internal nodes in the tree. For each node c in the tree, we denote $L(c)$ the sequence of beads corresponding to the leaves in the subtree rooted at c . This defines the canonical sub-necklaces of the full necklace in a standard fashion. As we will show later, the property of necklaces guarantees the efficiency of using such hierarchies in collision detection. Note that this is one instance where we heavily use the *a priori* known structure of the type of object we are modeling.

Given that our atomic elements are themselves balls, we have chosen to use spheres as our bounding volumes. Spheres do not bound as tightly as oriented bounding boxes (in the limit they have linear as opposed to quadratic convergence to an underlying smooth

shape [12]), but intersection/containment tests among them are especially simple and their symmetry makes rigid motions straightforward to implement. We define the *wrapped hierarchy* of a necklace to be the sphere hierarchy corresponding to the balanced tree described above, where the sphere corresponding to each internal node is the *minimum enclosing sphere* (MES) of the beads in the canonical sub-necklace associated with that node. We call these bounding spheres corresponding to internal nodes *cages*. Note that this allows the cages of the children of a node in the hierarchy to stick out of the cage of the parent. We call the sphere hierarchy defined by making the cage of a parent to be the MES of the cages of its two children the *layered hierarchy* [21]. Though the wrapped hierarchy is slightly more difficult to compute than the layered hierarchy, it is tighter fitting and most importantly it can be maintained more easily under deformation — a fact that at first seems counter-intuitive. An example of each type of hierarchy is shown in Figure 1. The key

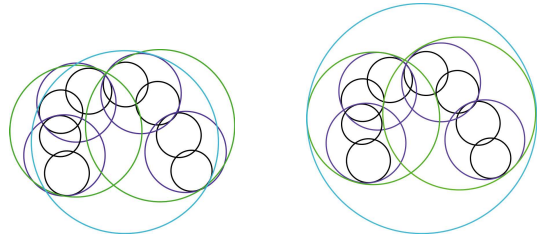


Figure 1: Wrapped (left) and layered (right) sphere hierarchies. The base beads are black. Notice that each cage in the wrapped hierarchy is supported by 2 or 3 beads.

property of the wrapped hierarchy that is of interest to us is that each cage, being a minimum enclosing sphere of a set of beads, is fully determined by a small number (two, three, or four in 3-D) of the beads in the associated canonical sub-necklace; these beads are called the *basis* of the cage and each of them is a *basis bead*. Note that the cage of an internal node is also the minimum enclosing sphere of its basis beads. When a necklace deforms, the basis of each cage remains constant for a period. At certain discrete events the basis of a cage changes typically by a pivoting step in which (1) an old basis bead leaves the basis, and (2) a new bead from the enclosed sub-necklace enters the basis. At times only one of these events may happen, but the total number of basis beads will always remain between two and four. Thus, although during continuous necklace deformation the cages deform continuously, their combinatorial descriptions stay constant and change only at discrete events. This combinatorialization of a continuous phenomenon is an insight analogous to what is exploited in kinetic data structures.

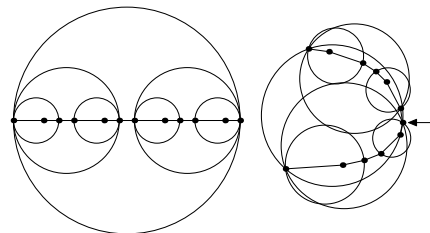


Figure 2: A combinatorially defined sphere hierarchy is stable under deformation. Only the top level cage differs between the two conformations.

We expect that under smooth deformation the combinatorial description of the cages via their basis beads will stay constant for

a fairly long time, and when finally the basis of a cage needs to change, that change will be easy to detect and the *basis update* simple to perform. For instance, in Figure 2 we show a 2-D example of such a hierarchy in two quite different configurations of a deforming necklace. It so happens that all the hierarchy cages except for the root cage continue to have the same combinatorial description at all intermediate configurations.

While the wrapped hierarchy is always tighter than the layered hierarchy, it is interesting to know exactly how much difference there can be between the two. In the following, we consider the case where all the beads are points (or equivalently equal radius spheres), and in arbitrary position. We have the following result:

THEOREM 4.1. *For any given set of n points in any dimension and any binary tree with depth $\lceil \log_2 n \rceil$ on the points, if we denote by τ_1, τ_2 the radii of the root spheres for the wrapped and layered hierarchies of the point set, respectively, then $\tau_2 \leq \tau_1 \sqrt{\lceil \log_2 n \rceil}$. The bound is almost perfectly tight, as we can construct a set of points so that $\tau_2 \geq \tau_1 \sqrt{\lceil \log_2 n \rceil}$.*

We denote the minimum enclosing sphere (MES) of a set of beads S by $M(S)$ and the MES corresponding to a node t by $M(t)$. The upper bound result is implied by the following lemma.

LEMMA 4.2. *Let O and R be the center and the radius of $M(S)$, and O_ℓ and R_ℓ the center and the radius of a ball on level ℓ in the layered hierarchy of S . Let $d_\ell = |OO_\ell|$. Then $R_\ell^2 \leq \ell(R^2 - d_\ell^2)$.*

Proof. Without loss of generality, let us assume that $R = 1$. We prove the lemma using induction. The lemma is clearly true for the 0-th level, as $R_0^2 \leq (1 - d_0)^2 \leq 1 - d_0^2$. We assume that the lemma holds for all balls at level ℓ in the layered hierarchy and show that the lemma still holds for balls at level $\ell + 1$.

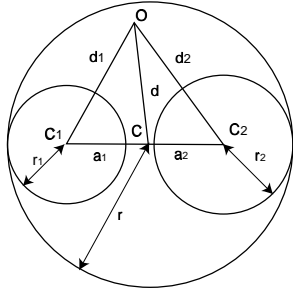


Figure 3: A ball in the layered hierarchy and its two children. Lemma 4.2 proves that the farther the centers of the children are from the center of their parent, the smaller their radii must be in comparison.

Let C and r be the center and the radius of a ball at level $\ell + 1$ in the layered hierarchy, and let C_1, C_2 and r_1, r_2 the centers and radii of its two children. Let $d = |OC|$, $d_1 = |OC_1|$, and $d_2 = |OC_2|$, see Figure 3. By induction assumption, $r_1^2 \leq \ell(1 - d_1^2)$ and $r_2^2 \leq \ell(1 - d_2^2)$. We would like to show that $r^2 \leq (\ell + 1)(1 - d^2)$. This is clearly true when the parent ball at C is identical to one of its children balls. We consider the general case when the parent ball is bigger than both of its children. The proof exploits the geometric fact that

$$d^2 = \frac{a_2 d_1^2 + a_1 d_2^2}{a_1 + a_2} - a_1 a_2. \quad (1)$$

The details can be found in Appendix A. \square

The above lemma immediately implies the upper bound in Theorem 4.1 as the depth of the tree is bounded by $\lceil \log_2 n \rceil$. Furthermore, we can show that the inequality in Lemma 4.2 can be made tight, and we can construct a set of points to attain the upper bound.

LEMMA 4.3. *There is a set of n points in the plane such that $\tau_2 \geq \tau_1 \sqrt{\lceil \log_2 n \rceil}$, where τ_1, τ_2 denote the radius of the root ball in the wrapped and layered hierarchy, respectively.*

Proof. It suffices to consider the case $n = 2^k$. We will construct a collection of points such that their wrapped hierarchy has radius 1 and their layered hierarchy has radius \sqrt{k} . The construction is done incrementally. We first fix any point O and place a point at O_0 such that $|OO_0| = 1$.

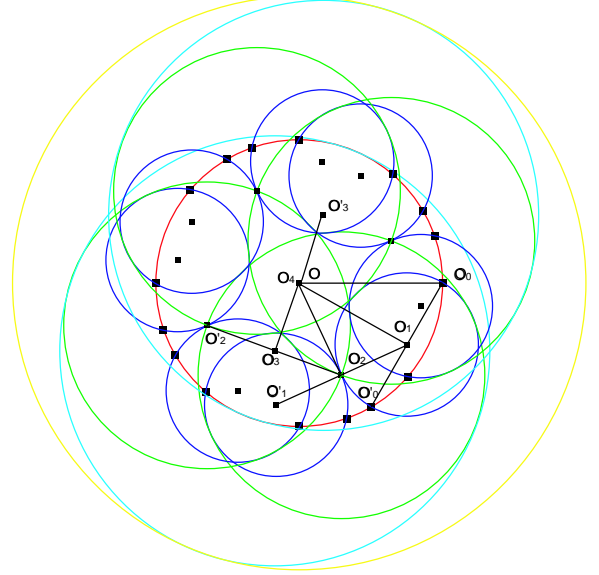


Figure 4: The construction of the $\sqrt{\lceil \log_2 n \rceil}$ worst case ratio for radius of layered hierarchy versus the wrapped hierarchy for $n = 16$.

Suppose that we have constructed the set S_ℓ , the first 2^ℓ points. Suppose that the root ball in the layered hierarchy of S_ℓ is centered at the point O_ℓ . To construct the set $S_{\ell+1}$, we first find the point $O_{\ell+1}$ such that (1) $\angle OO_{\ell+1}O_\ell = 90^\circ$, and (2) $|O_\ell O_{\ell+1}| = \frac{1}{\sqrt{k}}$. We can then construct $S_{\ell+1}'$ by flipping all the points in S_ℓ about the line $OO_{\ell+1}$. Then we set $S_{\ell+1} = S_\ell \cup S_{\ell+1}'$. See Figure 4.

It is straightforward to show that in the above incremental construction, the radius of the root ball of S_ℓ is $\frac{\ell}{\sqrt{k}}$, and $|OO_\ell| =$

$\sqrt{1 - \frac{\ell}{k}}$. Thus we can always find the point $O_{\ell+1}$, as long as $\ell < k$. It is clear that the root ball of the wrapped hierarchy has unit radius as all the points constructed are on the same circle, and the root ball of the layered hierarchy has radius \sqrt{k} .

Theorem 4.1 is the combination of Lemmas 4.2 and 4.3. \square

4.2 Construction and maintenance

It is straightforward to construct the wrapped hierarchy by directly computing the minimum enclosing ball for the canonical set of each cage. There is a complex algorithm for computing MES in $O(n)$ time in the worst case [19] and a simple one running in expected $O(n)$ time [24]. Therefore, it takes $O(n \log n)$ time to construct initially the wrapped hierarchy of the necklace of n beads.

To maintain the wrapped hierarchy, we need to verify the correctness of the hierarchy, i.e. the correctness of the basis beads at each internal node after a simulation time step, and update those which are no longer correct. To verify the validity of the basis beads, we make the following observation.

LEMMA 4.4. *Let \mathcal{B} and \mathcal{S} be sets of beads with $\mathcal{B} \subseteq \mathcal{S}$. If all beads $B \in \mathcal{S}$ are contained in $M(\mathcal{B})$, then $M(\mathcal{S}) = M(\mathcal{B})$. Thus, the basis for $M(\mathcal{S})$ is the same as the basis for $M(\mathcal{B})$.*

Note that the basis can shrink, so if \mathcal{B} was the basis of $M(\mathcal{S})$ before an update, and after the update $M(\mathcal{S}) = M(\mathcal{B})$, then some (not necessarily proper) subset of \mathcal{B} is the new basis for $M(\mathcal{S})$.

The checking is done in a hierarchical manner; we call this method the *cascade verification*. Suppose that we have checked the validity of the descendant nodes under a node c . We first compute the minimum enclosing sphere M of the basis beads from the previous time step. Lemma 4.4 proves that in order to check the correctness of c , it is sufficient to check that all the beads in $L(c)$, the leaf beads under the node c , are contained in M . To do this we maintain a frontier, F , initially containing the children of c . At each step we take a node d out of the frontier and check if $M(d)$ is contained in M . In case it is not, there are two cases — if the node is an internal node we add its children to F ; otherwise it is a leaf, so we know that it has escaped and the basis of node c is no longer valid and needs to be updated. If we can continue the above process until F becomes empty without encountering an escaped leaf node, we know that the basis for $M(c)$ is still valid. It is tempting to try to accelerate the above process by noting that the geometry belonging to a cage must be contained in the intersection of the cages on the path from that node to the root, and checking to see if this volume is contained in the cage being verified. However, in practice the extra complexity of this check more than outweighs its benefits. While in the worst case, the above procedure may take $\Theta(n \log n)$ time if all paths need to be traversed, our experiments suggest that in most instances the actual time is closer to linear, as only paths to the basis beads of the canonical subsequence need to be checked.

When beads escape from an enclosing cage c , a basis update must be performed. At least one of the escaped beads must be a basis bead for the new $M(c)$. The LP-type algorithm [24] allows easy exploitation of this knowledge in a natural manner, as well as easy use of other heuristic information about which beads are likely to be basis beads.

5. COLLISION AND SELF-COLLISION DETECTION

5.1 Separating beads by the wrapped hierarchy

Hierarchical representation is often used for collision detection between two objects or for self-collision detection. Typically, such an algorithm tries to compute a covering with *separating bounding volume pairs* by walking down the hierarchy. It reports a collision when it fails to produce such a covering. More precisely, suppose that B is a set of n disjoint objects $\{B_1, B_2, \dots, B_n\}$ and T is a balanced binary tree built on the elements of B . Just as before, for each internal node t in T , we create a bounding volume for the union of the objects which are the leaves in the subtree rooted at t . This way, we create a collection P of $2n - 1$ objects, among which n are basic objects and $n - 1$ are bounding volumes. A set C of pairs $\{(c_i, c_i')\}$, where $c_i, c_i' \in P$, is called a collection of separating pairs for B if:

1. for any i , c_i and c_i' are disjoint, and

2. for any i, j , where $i \neq j$, there exists k so that $B_i \in c_k$ and $B_j \in c_k'$.

Such a separating pair set serves as a proof of the non-collision of the set B , i.e. it exists if and only if all the objects in B are mutually disjoint. The minimum number of pairs needed to separate a set of objects is crucial as it is a lower bound of the cost of a collision detection algorithm taking such an approach. In our problem, the objects are beads and the bounding volumes are cages, and the hierarchical structure is the wrapped hierarchy. We also need to relax requirement 2 to $i \neq j - 1, j, j + 1$ as we allow adjacent beads to intersect each other. There has been some prior research on separating a set of balls. When we are allowed to group balls arbitrarily, there always exist a set of $O(n)$ separating pairs for n disjoint balls ([16, 5]). However, to our knowledge, the separating pairs for predefined hierarchical structures have not been studied combinatorially before. Here, we will show that for the wrapped hierarchy in d dimensions, there always exists a separation collection with $O(\max\{n \log n, n^{2-2/d}\})$ pairs if there is no collision between any pair of two non-adjacent beads. In the following, we do not distinguish a node in the tree and the cage built for that node.

THEOREM 5.1. *Let $\mathcal{B} = \{B_1, B_2, \dots, B_n\}$ be a sequence of n beads in d -dimensions (for $d \geq 2$), satisfying the uniformity and limited overlap assumptions. Then there exists a separating family for \mathcal{B} of size $O(\max\{n \log n, n^{2-2/d}\})$ among the cages in its wrapped hierarchy.*

Proof. By the uniformity and limited overlap assumption, we have that there exists a constant $\gamma \geq 1$ such that the distance between the centers of two adjacent beads is in the interval $[1/\gamma, \gamma]$. We give an algorithm for constructing a separating family Σ . Throughout the proof, \log is understood as \log_2 .

Fix an integer i such that $0 \leq i \leq \log n - 1$. Set $r_i = 2^i$. Let D_j be the cage in the wrapped hierarchy that encloses the beads $B_{(j-1)r_i+1}, \dots, B_{jr_i}$. Clearly, the radius of D_j is at most $R_i = r_i \gamma + \alpha$. Let $\mathcal{D}^i = \{D_1, D_2, \dots\}$ be the resulting set of balls. For each $D_j \in \mathcal{D}^i$, let K_j be the set of points that are distance at most $8R_i$ from a point in D_j , i.e., $K_j = \{x \mid \exists y \in D_j \quad |xy| \leq 8R_i\}$. K_j is a ball with radius at most $9R_i$ and concentric with D_j . For any ball $D_k \in \mathcal{D}^i$ that is contained in K_j and is disjoint from D_j , we report the pair (D_j, D_k) . We define Σ_j^i to be the set of pairs reported for D_j . We repeat this process for all balls in \mathcal{D}^i and set $\Sigma^i = \bigcup_j \Sigma_j^i$. Set $\Sigma = \bigcup_{i=0}^{\log n - 1} \Sigma^i$.

We claim that Σ is a separating family for B . It is obvious from the construction that all the pairs in Σ are disjoint. We need to argue that it covers all the pairs of beads. Consider a pair of disjoint beads (B_ℓ, B_m) . Denote by D_ℓ^i and D_m^i the i -th level (the level increases bottom up starting from 0) cages that contain B_ℓ and B_m , respectively. Let $t = \max \{i \mid D_\ell^i \cap D_m^i = \emptyset\}$. It is easy to see that every point in D_ℓ^t is within distance $4R_{t+1} \leq 8R_t$ from the center of D_m^t because D_ℓ^{t+1} and D_m^{t+1} intersect. Therefore $(D_\ell^t, D_m^t) \in \Sigma$. Hence, Σ is a separating family.

Next, we bound the size of Σ . Note that

$$\sum_{i=\frac{1}{2} \log n + 1}^{\log n - 1} |\mathcal{D}^i| = O(n^{1-1/d}),$$

because each ball in this set corresponds to the cage at a node in the wrapped hierarchy of B whose depth is less than $(1/d) \log n$.

Hence,

$$\sum_{i=\frac{1}{d} \log n+1}^{\log n-1} |\Sigma^i| = O(n^{2-2/d}).$$

It suffices to bound $|\Sigma^i|$ for $0 \leq i \leq (1/d) \log n$.

Every pair (D_j, D_k) in Σ_j^i ‘covers’ $r_i^2 = 2^{2i}$ pairs of beads. For any pair of beads (B_u, B_v) covered by this pair, their centers are within distance $9R_i$ because D_k is contained in K_j . Therefore, Σ^i covers $2^{2i}|\Sigma^i|$ pairs of beads with each pair is within distance $9R_i$. But by a packing argument, there are $O((9R_i\gamma)^d) = O(2^{di})$ beads whose centers are within distance $9R_i$ from the center of B_u . Hence, $2^{2i}|\Sigma^i| = O(n2^{di})$, which implies that $|\Sigma^i| = O(n2^{(d-2)i})$. Therefore,

$$\begin{aligned} |\Sigma| &= O(n^{2-2/d}) + \sum_{i=0}^{\frac{1}{d} \log n} O(n2^{(d-2)i}) \\ &= O(\max\{n \log n, n^{2-2/d}\}). \end{aligned}$$

This completes the proof of the theorem. \square

Remark. The above bound is tight in the worst case. Consider an $n^{1/d} \times \dots \times n^{1/d}$ d -dimensional grid. We can form a necklace B by tracing and connecting the segments parallel to the x -axis in the grid. By the construction, B contains $n^{1-1/d}$ x -axis aligned segments, each with length $n^{1/d}$. Any separating family of B in its wrapped hierarchy has size $\Omega(\max\{n \log n, n^{2-2/d}\})$. For two parallel segments at distance δ where $\delta \in [i, i+1)$, we need $\Theta(n^{1/d}/i)$ pairs to cover the beads on them. According to the bounds on the number of lattice points in circles, we know that there are $\Omega(n^{1-1/d}i^{d-2})$ such pairs of segments. So the number of pairs in any separating family is

$$\sum_{i=1}^{n^{1/d}} \Omega(ni^{d-3}) = \Omega(\max\{n \log n, n^{2-2/d}\}).$$

From Theorem 5.1, it follows that there always exists a collection of sub-quadratically many ($O(n \log n)$ in two dimensions and $O(n^{2-2/d})$ in d -dimensions for $d \geq 3$) separating pairs if any two non-adjacent balls are disjoint. The above constructive proof also suggests the following simple heuristic for collision detection using wrapped hierarchies: when two cages intersect, we always split the one containing more beads and break ties arbitrarily. This way, the cages we compare always have similar number of beads inside them, and their parent cages intersect. Therefore, the above proof applies — the number of pairs examined by the algorithm is bounded by Theorem 5.1. That is, such a collision detection algorithm takes time $O(n \log n)$ in two dimensions and $O(n^{2-2/d})$ in higher dimensions, in particular, it is $O(n^{4/3})$ in three dimensions. To our knowledge, this is the first guaranteed sub-quadratic bound for collision detection algorithms using bounding volume hierarchies.

5.2 Collision detection with the power diagram

Theorem 5.1 gives us a sub-quadratic bound on the running time of the collision detection method using the wrapped hierarchy. While the bound is $O(n \log n)$ in two dimensions, it is $\Theta(n^{4/3})$ in three dimensions. We observe that the wrapped sphere hierarchy is good for self-collision detection of a deforming necklace when the necklace is not too entangled. When the string of beads is highly packed, however, many cages in the hierarchy overlap with each other. We then have to traverse deeply down the hierarchy before being able to locate disjoint cages. According to a recent result by Jeff Erickson [10], the Delaunay triangulation has linear complexity for a

dense set of points. Although that result does not directly apply to the power diagrams, we may still expect that a similar result holds for the power diagram of a dense set of balls with comparable sizes. It was shown in [15] that the closest pair of balls are neighbors in the power diagram if all the balls are disjoint, and therefore we may use the power diagram for collision detection of a set of disjoint balls. In our problem, however, consecutive beads may overlap and that result no longer applies. In the following, we first show that we can still use the power diagram to perform collision detection, even when we allow some partial ball overlaps. Then, we discuss briefly how to maintain the power diagram under our motion model.

We prove our result about the power diagram in a more general setting. Let \mathcal{B} be a collection of balls and P be a set of pairs of balls. \mathcal{B} is disjoint with respect to P if every pair of balls are disjoint, unless it is in P . For example, a necklace is disjoint with respect to the set $P = \{(B_i, B_{i+1}) \mid 1 \leq i \leq n-1\}$. We define the closest pair of \mathcal{B} to be the pair with the minimum Euclidean distance among all the pairs not in P .

For any given P , a ball A is a neighbor of B if the pair (A, B) is in P . A ball A is a neighbor of a pair (B, C) if A is a neighbor of either B or C . A ball B is called a *proper connector* if for every neighboring ball A of B , there is another neighbor C of B such that A and C are disjoint. Balls that are not proper connectors are called *improper*. Note that when the balls are disjoint, there are no improper balls. For a necklace, only the first and last beads are improper. We then have the following result.

THEOREM 5.2. *The closest pair of \mathcal{B} must either (1) have an improper ball as a neighbor, or (2) have a common neighbor, or (3) share an edge in the power diagram of \mathcal{B} .*

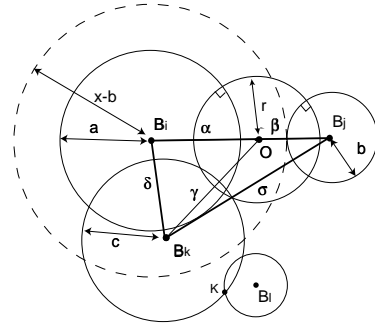


Figure 5: The power diagram can be used for collision detection in necklaces. The ball B_k cannot intersect ball O more than orthogonally, certifying the power diagram edge $B_i B_j$. x is the distance between B_i and B_j .

Proof. Suppose that (B_i, B_j) is the closest pair of balls in \mathcal{B} with respect to P . Assume that (B_i, B_j) does not satisfy (1) and (2). We would like to show that $B_i B_j$ is an edge in the power diagram of \mathcal{B} .

Let O be the minimum ball orthogonal to both balls B_i and B_j , and let r, a, b be the radii of O, B_i , and B_j respectively (Figure 5.2). Consider another ball B_k , where $k \neq i, j$, in \mathcal{B} . We would like to show that B_k does not intersect O more than orthogonally. According to [15], it is sufficient to consider those balls intersecting either B_i or B_j . Since B_i and B_j do not share common neighbor, we assume, without loss of generality, that B_k intersect B_i but disjoint from B_j . Let r be the radius of B_k . We denote the length of the line segments $OB_i, OB_j, OB_k, B_i B_k, B_j B_k$ by $\alpha, \beta, \gamma, \delta$, and σ , respectively, and the length of $B_i B_j$ by $x = \alpha + \beta$. We list the following conditions those quantities have to satisfy:

1. $\alpha^2 = r^2 + a^2$, and $\beta^2 = r^2 + b^2$. (by orthogonality)
2. $\sigma \geq x - a + c$. (by that the distance between ball B_i and ball B_j should be no more than the distance between ball B_j and ball B_k .)
3. $\delta \geq c - a$. (by triangle inequality $\delta \geq \sigma - x$, and condition 2.)
4. $\delta \geq x - b - c$. (since B_k is proper, there is another neighbor ball B_l of B_k so that B_i, B_l are disjoint, and so the distance between ball B_i and ball B_l is no less than the distance between ball B_i and ball B_j . If K is the intersection of balls B_k and B_l , then $\delta + c \geq |B_i K| \geq x - b$.)

Given the above relations, we would like to derive that the ball B_k does not intersect ball O' more than orthogonally, i.e. $\gamma^2 \geq c^2 + r^2$. By using Equation 1 and the above conditions, we can show that $\gamma^2 \geq c^2 + r^2$ and therefore complete the proof. The details can be found in Appendix B. \square

Theorem 5.2 gives us a way to check self-collision for a necklace: we can first compute the power diagram of all the beads and then check every pair for each power diagram edge. In addition, we check those pairs which share common neighbors, i.e. pairs (B_i, B_{i+2}) for $1 \leq i < n - 1$, and those pairs involving B_2 or B_{n-1} (the only beads having an improper neighbor in the necklace). Clearly, the number of additional pairs is $O(n)$, and the cost of the checking is dominated by the complexity of power diagram, which we expect to be linear for a densely packed necklace.

Under the KDS motion model, it is easy to maintain the power diagram [14]. For the discrete time step model, however, it can be too expensive to recompute the power diagram at each time step. We can imagine an incremental method where only the invalid parts of the diagram are repaired locally. Another way is to utilize the maintenance method intended for the continuous motion model. We can create a simple artificial motion to morph each bead in the previous time step to its new position at the current time step. We then run the kinetic algorithm for this imaginary motion and effectively morph the power diagram of the previous time step to the power diagram for the beads at the current position. Since the motion is artificial, we can design the motion so that the certificate failure times are easy to compute, or the number of events is small. For example, we can continuously change the size of beads so that each bead moves on a straight line in the standard lifting space. Due to lack of space, we do not elaborate on these options here.

6. EXPERIMENTAL RESULTS

We ran a variety of experiments to test the static properties of the wrapped hierarchy as well as its stability under motion. All the timings were done on a 700Mhz PIII and use Bernd Gärtner's [11] implementation of Welzl's algorithm for computing minimum enclosing spheres. The properties of interest were:

1. the cost of construction, as measured by the time taken,
2. the cost of verifying the hierarchy, using both the naive method (direct verification of bead inclusion in the cages) and the cascade verification, measured by both the time taken and the number of sphere inclusion tests performed,
3. the cost of collision detection in terms of time, the size of the set of separating pairs, and the number of intersection tests performed, and
4. the frequency of basis changes as the shape deforms.

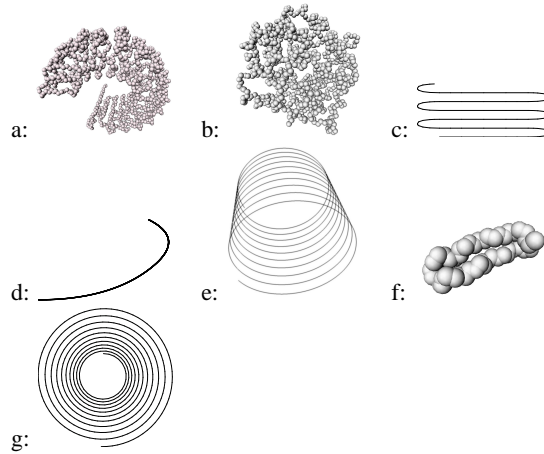


Figure 6: The necklaces used for testing. (a) is the backbone of the protein 1a4yA0, (b) is the backbone of the protein 1cem00, (c) is a 2-D grill which is a bad case for self-collision detection, (d) is a spline sampled at a variety of resolutions, (e) is a helix which has a quadratic complexity power diagram, (f) is a beta hairpin backbone fragment, and (g) is a 2-D spiral. (f) and (g) were also used to test stability under motion.

A variety of static curves were used for testing the first three properties. Two of them, (a) and (b), are protein backbones (1a4yA0 and 1cem00) consisting of slightly over 1000 atoms, each of whom was treated as having a radius of as 2\AA . The first protein has a great deal of secondary structure and is horse-shoe shaped, while the second is basically globular. The third, (c), is a 2-D grill-like curve which is a bad case for collision checking, according to Theorem 5.1. We also sampled a simple spline, (d), consisting of 2 quadratic segments, with 100 to 800 points. Our next shape (e) was a helix. The helix has constant height and radius, but it has variable number of turns and sample points. We tried two cases: 'static' helices with 10 turns and with 100 to 12,800 sample points, and 'scaling' helices with n sample points and \sqrt{n} turns, where n varies between 100 and 25,000. This shape is a bad case for the power diagram [9].

We had two data sets for testing the properties under dynamics. The first, (f), was a 51 atom backbone fragment from a beta hairpin, simulated through 300 time steps using a molecular dynamics package for protein folding (Tinker) [20]. The initial configuration was with the backbone folded along itself, a situation which did not visibly change during the time steps considered. The radii of all the atoms were treated as 2\AA for the purposes of building the wrapped hierarchy. The other dynamic case was a straight line segment rolling up into a spiral over 100 frames. The final form is shown in (g). The spiral was evenly sampled at 500 points along its length.

The cost of construction depends solely on the number of beads used. It took approximately 42ms to build the wrapped hierarchy for a protein backbone with 1380 beads. In general, the time dependence on the number of beads agrees well with the expected $O(n \log n)$ cost.

Similarly, the naive verification cost is mostly independent of the geometry. In contrast, the cascade verification algorithm depends quite heavily on the necklace shape. As a consequence, which verification algorithm was faster varied from model to model. Cascade verification performed best when the curve was smooth compared to the number of samples, i.e. when the curve crossing most of the spheres was diametral and fairly straight. For example, the two al-

gorithms performed similarly for the simple spline, (*d*), when the sampling was at 100 points, but they diverged as the sampling increased, and cascade verification was twice as fast when (*d*) was sampled with 800 points. This high sampling was typical of the good cases for cascade verification in that the only descendant cages which stick out are those that contain the end-points of the canonical sub-necklace. As a result, the average verification cost per cage is constant and independent of the number of beads (since it is twice the average distance to the leaves of the tree).

Model	Nklc Size	S. Col. Tests	S. Col. Front.	Casc. Tests	Casc. Front.	Naive Tests
(a)	1380	9.8	5.7	7.6	5.8	13
(b)	1089	7.2	5.0	6.4	4.8	10
(c)	63	6.5	4.9	6.2	4.4	8.5
(d)	100	.98	1.4	7.0	4.2	8.6
	400	1.0	2	7.0	4.3	11
	1600	1.0	1.5	7.0	4.3	14
(e) static	400	9.7	7.2	6.1	15	3.9
	1600	3.5	3.3	14	7.7	14
	6400	1.7	2.0	13	6.9	16
(e) scaling	100	19	11	12	5.7	8.6
	400	28	22	16	8.1	11
	1600	41	32	19	10	14
	6.4k	166	130	23	12	16
	25k	672	530	28	15	19
(f)	51	6.0	3.3	5.6	3.5	7.2
(g), start	500	1.0	1.5	3.1	2.5	9.1
	end	500	8.1	5.7	8.1	4.6

Figure 7: “Nklc Size” is necklace size. “S. Col. Tests” is the average number of intersection tests for self-collision per hierarchy node, and “S. Col. Front.” is the average size of the frontier per node. “Casc. Tests” is the number of comparisons for cascade verification per node, and “Casc. Front” is the frontier size per node. Finally, “Naive Tests” is the average number of comparisons for naive verification, namely the average number of leaves per node.

At the other extreme, the naive algorithm was twice as fast as the cascade verification on the scaling helix (*e*). On such a tightly coiled shape, most child cages stick out from their parents, consequently, cascade verification must explore the whole subtree before getting to the beads, resulting in almost twice as many intersection tests. The other experiments were somewhere in between the two extremes, and the two algorithms were generally fairly equivalent in time and work. The minimum times for verification ranged from 6ms for the smooth curve sampled with 800 points to .8s for the scaling helix with 25,600 beads.

When the objects are deforming, we can save the ‘frontier’ from the earlier time step in the cascade verification algorithm and use that as a starting point for the verification in the next time step, allowing a smooth transition between the two algorithms. Even in the worst case of the scaling helix with 25,000 beads, the frontier was less than 16 descendant nodes per cage being tested.

Collision checking performance was tested using the self-collision algorithm. The collision checking algorithm is fairly similar to the verification algorithm, so it is not surprising that the good and bad cases were similar. The spline, (*d*), was the fastest to check, as the algorithm was able, in most cases, to simply verify that the two siblings did not intersect due to the small size of the beads compared to their separation. However, even if the beads did intersect,

there would still be an average constant cost per node for reasons similar to verification. The separating set scaled linearly with the sampling of (*d*), from 35 for a sampling of 100 beads to 575 for a sampling of 1600. Performance on the grill curve, (*c*) was as expected. While the separating set size for (*d*) sampled at 100 points was only 35, the set size for the grill was 309, and the verification took correspondingly longer, at .6ms for the grill.

As before, the scaling helix was far and away the worst case. Self collision on the 25,600 bead sampling took 18 seconds, and the 100 bead sampling took 2ms. The separating sets were similarly enormous, being almost 10 million pairs for the high sampling rate and 1000 pairs for the 100 bead sample. This set is too large to hope for gains from a caching scheme analogous to that proposed for the verification stage. However, this is a pathologically bad case, and in many applications such situations can be ruled out.

A more interesting case was the static helix. When sampled with 200 points, self-collisions took 3.25ms, involved 14 checks per node and had a separating set of 2000 pairs. As the sampling rate increased, the number of checks per node decreased. At the highest sampling rate with 12,800 beads, it took 20ms and only involved 1.3 intersection tests per cage and the separating set was only 6,800 pairs. The reason for this is that as the sampling increases, amount of bending per bead decreases so the curve looks straighter and straighter on the scale of most of the spheres. As a result, a greater fraction of the collision tests became trivial ones as in the smooth curve case.

The wrapped hierarchy was very stable under the deformations of the underlying necklace in our tests. For the hairpin simulation (*f*), the cascade verification performed similarly to the naive verification, both averaging approximately 7 containment tests for each of the 37 nodes, and the separating set had a size of about 30. There were 19 basis changes over the 300 frames of the simulation. For comparison, we also built the power diagram and looked at its stability. The power diagram was much more complex with approximately 300 edges and 450 faces. The total number of changes was close to 1500.

For the spiral tests set (*g*), the wrapped hierarchy had 487 cages. Since the sequence involved a change from a straight line (an optimal configuration for our algorithm), to a spiral (a near worst case configuration), the wrapped hierarchy verification performance varied quite considerably. In the first frame, cascade verification was twice as fast as naive and made 3 comparisons per node, while in the last frame cascade verify was slightly slower and took 8 inclusion tests per cage. Self-collision fared even worse, with the separating set size going from 243 pairs to 2,868 pairs. There were a total of 500 basis changes. In comparison, the power diagram had 1000 edges in the first frame, and 498 faces, compared to 1,425 edges and 926 faces in the last frame. The total number of events in the power diagram was over 50,000.

The experiments show that the wrapped hierarchy is quite stable under deformation and efficient to maintain for elongated or relatively unpacked shapes, while the verification and collision detection become expensive when the curve is tangled and becomes globular. The cost of self-collision checking and hierarchy verification seems to depend on the amount of bending at each bead, but we have so far been unable to quantify the dependence. The experiments also show that the power diagram is more complex than the wrapped hierarchy for the cases we tested — while the verification and collision checking cost for the power diagram is low, the maintenance cost is significantly higher. When dealing with curves that are loosely packed, the greater simplicity of the wrapped hierarchy clearly wins; the verdict it is less clear with necklaces that are more tightly packed. We can hope that a hybrid method exists that com-

bines the best features of the sphere hierarchy and power diagram approaches.

7. CONCLUSIONS

This paper raises a new set of issues in geometric computing, by posing the problem of how to repair and maintain geometric structures under small motions or deformations of their defining elements. Efficient geometric structure repair is essential in complex physical simulations, virtual reality animations, as well as when tracking real world objects. More generally, additional research is needed on how to better integrate geometric algorithms with physical models of objects.

Even for our simple example of a deforming necklace, many basic questions remain open:

- What are the trade-offs between the wrapped and layered hierarchies?
- Can we prove bounds on the number of combinatorial changes in the wrapped hierarchy, assuming a physical model of deformation and a given ‘deformation energy budget’ that limits the overall bending that can occur?
- how can we best integrate the power diagram and the sphere hierarchy so as to get the advantages of each?

We hope to address some of these issues in the near future.

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APPENDIX

A. PROOF OF LEMMA 4.2

Let $a_1 = |CC_1|$, $a_2 = |CC_2|$ as in Figure 3. Then $r = r_1 + a_1 = r_2 + a_2 = s + a$, where $s = \frac{r_1+r_2}{2}$ and $a = \frac{a_1+a_2}{2}$. Let $\delta = \frac{a_2-a_1}{2} = \frac{r_1-r_2}{2}$. It is easy to see that

$$\begin{aligned} a_1 &= a - \delta, & a_2 &= a + \delta, \\ r_1 &= s + \delta, & r_2 &= s - \delta, \\ d_1^2 &\leq 1 - r_1^2/\ell, & d_2^2 &\leq 1 - r_2^2/\ell \end{aligned}$$

From $\cos(\angle OCC_1) = -\cos(\angle OCC_2)$ and the law of cosines,

we obtain:

$$\begin{aligned}\frac{a_1^2 + d^2 - d_1^2}{2a_1d} &= -\frac{a_2^2 + d^2 - d_2^2}{2a_2d} \\ a_2(a_1^2 + d^2 - d_1^2) &= -a_1(a_2^2 + d^2 - d_2^2) \\ (a_1 + a_2)d^2 &= (a_2d_1^2 + a_1d_2^2) - (a_2a_1^2 + a_1a_2^2) \\ d^2 &= \frac{a_2d_1^2 + a_1d_2^2}{a_1 + a_2} - a_1a_2\end{aligned}$$

Thus,

$$\begin{aligned}d^2 &\leq \frac{(a + \delta)(1 - r_1^2/\ell) + (a - \delta)(1 - r_2^2/\ell)}{2a} \\ &\quad - \frac{(a - \delta)(a + \delta)}{2a} \\ &= \frac{2a - a(r_1^2 + r_2^2)/\ell - \delta(r_1^2 - r_2^2)/\ell}{2a} - (a^2 - \delta^2) \\ &= 1 - \frac{1}{2\ell}(r_1^2 + r_2^2) - \frac{\delta}{2a\ell}(r_1^2 - r_2^2) - (a^2 - \delta^2) \\ &= 1 - \frac{1}{2\ell}2(s^2 + \delta^2) - \frac{\delta}{2a\ell}(4s\delta) - (a^2 - \delta^2) \\ &= 1 - \frac{s^2 + \delta^2}{\ell} - \frac{2s\delta^2}{a\ell} - (a^2 - \delta^2)\end{aligned}$$

We would like to show that $r^2 \leq (\ell + 1)(1 - d^2)$.

$$\begin{aligned}&(\ell + 1)(1 - d^2) - r^2 \\ &\geq (\ell + 1)\left[\frac{s^2 + \delta^2}{\ell} + \frac{2s\delta^2}{a\ell} + (a^2 - \delta^2)\right] - (s + a)^2 \\ &\quad + \frac{s^2 + \delta^2}{\ell} + \frac{2s\delta^2}{a\ell} + (a^2 - \delta^2)\end{aligned}$$

by expanding and simplifying the expression, we get:

$$= \frac{2s\delta^2}{a} + \ell a^2 - \ell \delta^2 - 2as + \frac{s^2 + \delta^2}{\ell} + \frac{2s\delta^2}{a\ell}$$

by completing the square, we get:

$$\begin{aligned}&= (\ell a^2 - 2as + \frac{s^2}{\ell}) - (\ell \delta^2 - \frac{2s\delta^2}{a} + \frac{s^2\delta^2}{\ell a^2}) \\ &\quad + (\frac{s^2\delta^2}{\ell a^2} + \frac{2s\delta^2}{a\ell} + \frac{\delta^2}{\ell}) \\ &= \frac{(\ell a - s)^2}{\ell} - \frac{(\ell a - s)^2\delta^2}{\ell a^2} + \frac{(s + a)^2\delta^2}{\ell a^2} \\ &= \frac{(\ell a - s)^2(a^2 - \delta^2) + (s + a)^2\delta^2}{\ell a^2} \\ &= \frac{(\ell a - s)^2 a_1 a_2 + (s + a)^2 \delta^2}{\ell a^2} \\ &\geq 0\end{aligned}$$

Thus, we have that $r^2 \leq (\ell + 1)(1 - d^2)$. \square

B. PROOF OF THEOREM 5.2

From Equation 1 in the proof of Lemma 4.2, we have

$$\gamma^2 = \frac{\beta\delta^2 + \alpha\sigma^2}{x} - \alpha\beta.$$

We substitute $x - a + c$ for σ and the larger of $c - a$ or $x - b - c$ for δ to find a lower bound for γ^2 .

1. When $c - a \geq x - b - c$, i.e. $c \geq (x + a - b)/2$, we obtain

$$\begin{aligned}\gamma^2 &\geq \frac{\beta(c - a)^2 + \alpha(x - a + c)^2}{x} - \alpha\beta \\ &= c^2 + 2(\alpha - a)c + (\alpha - a)^2 \\ &\geq c^2 + (\alpha - a)(x + a - b) + (\alpha - a)^2 \\ &= c^2 + r^2 + (\alpha - a)(\beta - b) + (\alpha - a)^2 \\ &\geq c^2 + r^2.\end{aligned}$$

2. When $c < (x + a - b)/2$, similarly, we get

$$\begin{aligned}\gamma^2 &\geq \frac{\beta(x - b - c)^2 + \alpha(x - a + c)^2}{x} - \alpha\beta \\ &= c^2 + 2\frac{cr^2}{x} \left(\frac{\alpha}{\alpha + a} - \frac{\beta}{\beta + b} \right) \\ &\quad + r^2 \left(\frac{\alpha - a}{\alpha + a} + \frac{\beta - b}{\beta + b} \right) + r^2.\end{aligned}$$

If $\frac{\alpha}{\alpha + a} \geq \frac{\beta}{\beta + b}$, it is clear that $\gamma^2 \geq c^2 + r^2$. If not, using $c > (x + a - b)/2$, we have that

$$\begin{aligned}\gamma^2 &> c^2 + r^2 \frac{x + a - b}{x} \left(\frac{\alpha}{\alpha + a} - \frac{\beta}{\beta + b} \right) \\ &\quad + r^2 \left(\frac{\alpha - a}{\alpha + a} + \frac{\beta - b}{\beta + b} \right) + r^2 \\ &= c^2 + r^2 + \frac{\alpha - a}{\alpha + a} r^2 \\ &\quad + \frac{\alpha(\beta^2 - b^2) + \beta(\alpha^2 - a^2)}{(\alpha + \beta)(\alpha + a)(\beta + b)} r^2 \\ &\geq c^2 + r^2.\end{aligned}$$

In both cases, $\gamma^2 \geq c^2 + r^2$. \square